# SOME STUDIES IN GAS-PARTICULATE FLOW PROBLEMS

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*By*

SAROJ PRABHA

*to the*

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*To My Father*

CERTIFICATE

This is to certify that the work embodied in the thesis "Some Studies in Gas-Particulate Flow Problems" by Saroj Prabha has been carried out under our supervision and has not been submitted elsewhere for a degree or diploma.

(A.C. Jain)
Professor
Department of Aeronautical Engineering
Indian Institute of Technology
Kanpur-208016

(R.K. Jain)
Professor
Department of Mathematics
Indian Institute of Technology
Kanpur-208016

January-1980

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Saroj Prabha

*January-1980* *Saroj Prabha*

CONTENTS

## SYNOPSIS

In the present investigation, an attempt is made to understand the basic nature of gas-particulate flows under various situations. Various problems that generally arise in Aeronautics, Industrial and Bio-fluid mechanics and natural aerodynamics are considered. Mathematical models based upon Navier-Stokes equations or its truncated form viz. boundary layer equations are considered. Particulate phase being pseudo fluid lacks definite boundary condition on the surface. A discussion of the various boundary conditions is undertaken. Various analytical techniques and numerical methods are used to solve these equations. Finally the global stability and uniqueness criteria of an IBVP (initial boundary value problem) of a dusty gas model are discussed.

In Chapter I, basic assumptions involved in a mathematical formulation of a gas-particulate flow problem are stated and a critical assessment of the governing equations namely Navier-Stokes equations and the derived boundary layer equations is included. A description of various boundary conditions on the surface for particulate phase is also incorporated.

In Chapter II, gas-particulate boundary layer equations valid for flows on a flat plate are integrated numerically with and without the use of compatibility conditions on the surface. It is found that the surface characteristics in either case

remain the same but the detailed structure of the flow in the boundary layer is predicted better when the compatibility conditions are dropped from the analysis.

In Chapter III, the problem related to the gas-particulate flows in between porous parallel plates with moving upper surface **is** considered. Both steady and unsteady problems are dealt with. Closed form solutions using various mathematical techniques are derived. This study enhances our understanding of the basic nature of the flow in certain simple situations.

In Chapter IV, the gas-particulate flow in tubes of varying cross-section is considered. Here, an asymptotic series solution of Navier-Stokes equations is developed. Three terms of the series are evaluated analytically. Zeroth order term gives the Poiseuille flow. Higher order terms give correction due to varying cross-section. For a diverging channel, the particles move towards the wall. Shear stress at the wall decreases from its value for gas phase when particles are present. The present analysis is essentially valid for low Reynolds numbers. In case, we compute the flow for high Reynolds number, eddies are formed near the wall.

In Chapter V, the problem of gas-particulate flow through curved pipes is solved. Dean (Mathematika, 1959, Vol. 13, p.77) discussed the motion of the gas phase in the curved pipe and developed analytical solution by using the method of separation of variables. This analysis is extended to gas-particulate

flows and the method of Green's function is used to solve the governing equations. The problem is solved for pipes of rectangular and circular cross-sections. It is found that the addition of fine dust particles decreases the volume flow rate while the presence of coarse dust particles increases the volume flow rate.

In Chapter VI, the global stability and uniqueness of an IBVP of a dusty gas model are discussed. Two types of uniqueness theorems of an IBVP for Navier-Stokes equations are pointed out. First of them ensures only one solution for each initial data while second guarantees only one steady state solution for not too small a viscosity.

# CHAPTER I

# INTRODUCTION

## 1.1 General Introduction

An understanding of the basic phenomenon in gas-particulate flows is essential in Natural and Industrial Aerodynamics, several branches of engineering, environmental, biological and physical sciences etc. In two phase flow problems, suspended matter may consist of solid particles, liquid droplets, gas bubbles or some combination of these. In environmental sciences, we are greatly concerned with the dispersion and fall out of pollutants in air and the problem may be related to the nuclear fall out. Some of the other areas when the problem under consideration may find its applications are : movement of dust laden air, fluidization; use of dust in gas cooling systems to enhance heat transfer processes; coalescence of small droplets to form raindrops which might be considered as solid particles for the purpose of examining their movement; nuclear reactors with gas solid feeding; ablation cooling; batch settling; powder technology; dust collectors; sedimentation; acoustics; aerosol and paint spraying; blood flow with red cell particles; pneumatic transport; slurry pipe lines; erosion of material due to continuous impingement of suspended particles in air etc.

During the past two decades considerable progress has been made in the development of continuum theories of particulate suspension. In a continuum model, the particulate phase is considered as a cloud of particles. This cloud of particles is called pseudo fluid. The concept of pseudo fluid enables us to deduce the governing equations for gas-particulate flows similar to Navier-Stokes equations for pure gas phase. In the present thesis, gas-particulate flows, flows with suspension and two phase flows have the same connotation.

In § 1.2, we derive the governing equations for the two phase flow considering the gas and particle phases as compressible. Here, the terminology of compressibility for the particulate phase has been used only to indicate that the particulate density is a variable i.e. it is a function of space and time coordinates. Soo (1967) has called it as special compressibility. We also deduce here the corresponding energy equations for the two phases. In order to gain some basic understanding of the underlying physical phenomenon of the two phase flows, laminar gas-particulate flows are considered in the thesis. Hence, the governing equations are also derived for laminar flows.

In § 1.3, we have deduced the boundary layer equations when the particle phase **is** compressible and there is negligible volume fraction of the particles. In § 1.4, a discussion of

the boundary conditions is included. Special attention is given to the derivation of compatibility conditions for particulate phase , its physical interpretation and its drawbacks. Various forms of the governing equations so deduced are used in solving various physical problems in the following chapters. In § 1.5, a brief description of the various problems solved in the thesis, the method of solution and the important results obtained is included.

## 1.2 Discussion of the governing equations of motion of two phase flows

Several investigators have derived the basic equations of the two phase flows. Significant contributions in this direction had been made by Soo (1967), Marble (1962), Pai (1970), Murray (1965), Van Deemter and Van Der Laan (1961), Drew and Seigal (1971) and Hinze (1963). Marble (1962) used the kinetic theory approach to derive the governing equations of motion. Van Deemter and Van Der Laan (1961) derived the continuity, momentum and mechanical energy equations in a formal way. They did not discuss the stress tensor and interaction forces. Conceptually their equations add little to our understanding of the basic nature of the flow of suspensions. Murray (1965) discussed the equations for two phase flows by considering the contributions of shear stresses in both the phases separately. His volume element of the mixture was made up

of fluid and particle volumes. Murray (1965) derived these equations for fluidized beds but these are valid for dispersed two phase flows also. According to Drew and Seigal (1971), equations derived by Murray (1965) are not consistent as the equations are not valid for an arbitrary volume element of the fluid. Drew and Seigal (1971) derived the equations for two phase flows by averaging the various flow quantities over a given volume. These derivations are quite general and are valid for a large class of problems. In a later paper, Drew and Seigal (1971) applied the averaged equations to the fluidized beds and foams. Pai (1970) also used the phenomenological approach to derive these equations. He did not consider the effect of volume fraction of the particles on the stress tensor and the pressure gradient term.

Here, we discuss the equations given by Hinze (1963) for two phase flows. These equations contain the effect of volume fraction of the particles. Although in the thesis, we consider negligible volume fraction of the particles in different flow problems, but this procedure enables us to envisage clearly the various terms that may imbibe the effect of the presence of the particles. It gives a better feeling of the approximations that have been incorporated by taking negligible volume fraction of the particles in the flow. Also, it opens the possibility of further extension of the problems discussed in the thesis where the volume fraction of

the particles becomes an important feature of the physical problem.

Consider the fluid with velocity $u_g$, temperature $T_g$ and density $\rho_g$ with a cloud of solid spherical particles having the same radius 'a'. The particle cloud is also described by the corresponding continuum variables $u_p$, $T_p$ and $\rho_p$. Here $\rho_p = nm$ where n is the number of particles per unit volume and m is the mass of each particle. We assume that the particles are sparsely distributed and non-interacting. Flow around each particle is uneffected by the presence of other particles in the flow. Let $\alpha$ denotes the volume fraction of the particles in a unit volume, $\rho_{sg}$ and $\rho_{sp}$ the densities of the material of the gas and the particles respectively, then

$$\rho_g = (1-\alpha)\,\rho_{sg} \tag{1.1}$$

$$\rho_p = \alpha\,\rho_{sp} \tag{1.2}$$

The density $\rho_M$ and average velocity $U_M$ of the mixture are defined as follows:

$$\rho_M = (1-\alpha)\,\rho_{sg} + \alpha\,\rho_{sp} \tag{1.3}$$

$$U_M = (1-\alpha)\,u_g + \alpha\,u_p \quad \text{(Volume velocity)} \tag{1.4}$$

$$\rho_M U_M = (1-\alpha)\,\rho_{sg}u_g + \alpha\,\rho_{sp}u_p \quad \text{(Mass velocity)} \tag{1.5}$$

Equations of continuity and momentum for the fluid and particle phases are the following:

$$\frac{\partial}{\partial t}(\rho_g) + \frac{\partial}{\partial x_i}(\rho_g u_{gi}) = 0 \tag{1.6}$$

$$\frac{\partial}{\partial t}(\rho_p) + \frac{\partial}{\partial x_i}(\rho_p u_{pi}) = 0 \tag{1.7}$$

$$\frac{\partial}{\partial t}(\rho_g u_{gi}) + \frac{\partial}{\partial x_j}(\rho_g u_{gi} u_{gj}) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij}) - (1-\alpha)\,\rho_{sg}\, g\, \delta_{i\ell} + F_i \tag{1.8}$$

$$\frac{\partial}{\partial t}(\rho_p u_{pi}) + \frac{\partial}{\partial x_j}(\rho_p u_{pi} u_{pj}) = -\alpha\,\rho_{sp}\, g\, \delta_{i\ell} - F_i \tag{1.9}$$

Here, Einstein summation convention is used. In eqs. (1.8) and (1.9), $\ell$ is the direction parallel to the gravitational acceleration 'g' and $F_i$ represents the $i^{th}$ component of the iteraction forces due to virtual mass, buoyancy, skin-friction resistance due to particles and resistance due to pressure gradient in the fluid phase.

$$F_i = B\,\alpha\rho_{sg}\left(\frac{Du_{pi}}{Dt_p} - \frac{Du_{gi}}{Dt_g}\right) - g\,\alpha\rho_{sg}\,\delta_{i\ell} - R_i + \alpha\frac{\partial p}{\partial x_i} \tag{1.10}$$

where

$$\frac{D}{Dt_g} = \frac{\partial}{\partial t} + u_{gk}\frac{\partial}{\partial x_k} \tag{1.11}$$

$$\frac{D}{Dt_p} = \frac{\partial}{\partial t} + u_{pk}\frac{\partial}{\partial x_k} \tag{1.12}$$

and

$$\tau_{ij} = \mu_m\left(\frac{\partial U_{Mi}}{\partial x_j} + \frac{\partial U_{Mj}}{\partial x_i}\right) - \frac{2}{3}\,\mu_m\frac{\partial U_{Mk}}{\partial x_k}\,\delta_{ij} \tag{1.13}$$

$\mu_m$ being the viscosity of the mixture.

In eq. (1.8), Hinze (1963) has introduced the stress tensor for the gas-particulate flow which incorporates the effect of the volume fraction of the particles in the description of the viscosity and the gradients of the mixture volume flow velocity. Due to the non interacting nature of the particles, stress tensor of the particles is not present in the momentum equation (1.9) for the particle phase but due to the velocity of the particle phase, the mixture velocity is in general different from the velocity of either phase. Hence the deformation of the fluid element depends on the gradients of the mixture velocity. As is discussed earlier in the text, investigations of Murray (1965), Drew and Seigal (1971) show that the stress tensor can be expressed differently.

In the literature, various expressions for the coefficient of viscosity of suspensions $\mu_m$ had been given. Einstein (1906) gave the following expression for the viscosity of the gas-particulate mixture

$$\mu_m = \mu_g (1 + 2.5 \alpha) \tag{1.14}$$

where $\mu_g$ is the viscosity of the gas phase. Happel (1959) gave the following expression

$$\frac{\mu_m}{\mu_g} = 1 + 5.5 \alpha \tag{1.15}$$

This formula differs from (1.14) because Happel considered the interaction forces of the particles. Eqs. (1.14) and (1.15) show that the viscosity of the gas increases by adding dust

particles to it. This fact was also supported experimentally by Ward and Whitmore (1950) and Williams (1953).

Murray (1965) pointed out that static pressure in the particle phase is zero due to non interacting nature of the particles. Also, for a mixture of practical interest, the number density of particles is insignificant in comparison to the number density of the gas molecules. Therefore the contribution of the particles to the pressure of the mixture is negligible. Rudinger (1965) derived the following relation expressing the gas phase pressure in terms of the mass fraction of the particles

$$\frac{p_g}{p} = \left(1 + \frac{\phi}{1-\phi}\frac{\rho_{sg}}{\rho_{sp}}\frac{3L^{-1}}{4\pi a^3}\right)^{-1} \qquad (1.16)$$

where $p_g$ is the pressure due to the fluid phase, $p$ the pressure due to the mixture, $\phi$ the mass fraction of the particles and $L = 2.69 \times 10^{19}$ cm$^{-3}$ is the Loschmidt number. He found that in case, the contribution of the solid particles to the pressure of the mixture is to be less than one percent ($p_g/p \geq 0.99$), then

$$d \geq 0.0192 \left[\frac{\rho_{sp}}{\rho_{sg}}\frac{1-\phi}{\phi}\right]^{-1/3}, \quad d = 2a \qquad (1.17)$$

which shows that in most cases of interest, the particles contribution to pressure of the mixture is negligible if the particle diameter is less than a hundreth of a micron. He further suggested that the particles contribution to the pressure

can be neglected even at extremes of the density ratios.

For reasons stated above, equation of state is taken as

$$p = R\, \rho_g\, T_g \quad , \text{ R being gas constant} \tag{1.18}$$

The term

$$B \;\; \alpha\, \rho_{sg} \left( \frac{Du_{pi}}{Dt_p} - \frac{Du_{gi}}{Dt_g} \right)$$

in the interaction term (1.10) corresponds to the force due to added mass of the particles. The numerical factor B corresponds to the volume of the virtual mass of the fluid subjected to the acceleration of the particles relative to the ambient fluid. For a spherical particle in an unbounded region, B = 0.5. Here B may differ from this value because of the presence of the neighbouring particles.

Hinze (1963) did not add this term in the momentum equation (1.8) for the fluid phase but according to Newton's third law of motion, this term must also be added in the momentum equation (1.8) for the fluid phase. Murray (1965) pointed out that the form of this term is a conjecture. A term of the form

$$B\, \alpha\, \rho_{sg} \left[ \frac{\partial}{\partial t} (u_{p_i} - u_{g_i}) + (u_{p_k} - u_{g_k}) \frac{\partial}{\partial x_k} (u_{p_i} - u_{g_i}) \right.$$

can also be used.

Inspite of the fact that pressure is negligible due to the solid particles, the term $(-\alpha \frac{\partial p}{\partial x_i})$ in the solid phase is

introduced [Hinze (1963)] mainly due to the resistance it offers to the particle motion. There is considerable controversy about the nature and the form of the pressure gradient term in the two phase flow equations. Some of these points are resolved by Shah and Soo (1979) and Boure (1979).

$R_i$ is the drag force due to difference in velocity in fluid and particle phases. For the discussion of drag forces we shall consider the particles to be spherical in shape and of radius 'a'. For small $R_e$ (of order unity) Proudman and Pearson (1957) gave the following formula for the drag coefficient

$$C_D = \frac{24}{R_e} \left[ 1+ \frac{3}{16} R_e + \frac{9}{160} R_e^2 \log R_e + O\left(\frac{R_e^2}{4}\right) \right] \qquad (1.19)$$

where the leading term gives the Stokes law of resistance and the second term gives the correction due to the Stokes law by Oseen.

In the literature, a large number of resistance laws incorporating the effect of Mach number, solid particle temperature, turbulence and rarefaction effects are available. Recently, Henderson (1976) gave such an emperical law for the subsonic and supersonic regimes. His formula for subsonic regime is valid upto Mach number $M \leq 1.17$.

For the sake of simplicity and ease in analytical calculations, we have adopted the Stokes law of resistance in our analysis. While passing, we may mention that the

free stream Reynolds number is large but the Reynolds number based upon the particle diameter and the relative velocity of the particle and the fluid may be small. Hence, the Stokes law may represent the resistance to the fluid with reasonable accuracy. If there are n particles in a unit volume of the fluid, then the resistance to the flow is

$$R_i = n\, 6\pi\, \mu_g\, a(u_{gi}-u_{pi})$$

which can be rewritten as

$$R_i = \rho_p\, F(u_{gi}-u_{pi}) \tag{1.20}$$

where

$$F = \frac{18\,\mu_g}{d^2\rho_{sp}} \tag{1.21}$$

Other interaction forces may be the lift and a torque acting on the particle. These forces would cause the particle translation in the direction normal to the flow as well as particle rotation. Saffman (1965) calculated the lift force on a single particle. Rubinow and Keller (1961) gave an expression for the torque acting on a spherical particle. For small $R_e$, Hamed and Tabakoff (1975) gave the following expressions for the forces acting along and perpendicular to the flow and torque which incorporated the results of Saffman (1965), Rubinow and Keller (1961), Oseen and Tam (1969)

$$R_x = \frac{\alpha\rho_{sp}}{\tau_T}(u_g-u_p)\,\{1+1.5(2\alpha)^{1/2}+3.75\,\alpha\}$$
$$\{1+\frac{3}{16}\,\frac{d}{\nu}\,[(u_g-u_p)^2+(v_g-v_p)^2]^{1/2}\}$$
$$-\frac{9.69\,\alpha\,\mu_g}{\pi d}\,(v_g-v_p)\,[\frac{1}{\nu}\,(\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y})]^{1/2} \tag{1.22}$$

$$R_y = \frac{\alpha \rho_{sp}}{\tau_T} (v_g - v_p) \{1 + 1.5(2\alpha)^{1/2} + 3.75\alpha\}$$

$$\{1 + \frac{3}{16} \frac{d}{\nu} [(u_g - u_p)^2 + (v_g - v_p)^2]^{1/2} \}$$

$$- \frac{9.69\, \alpha \mu_g}{\pi d} (u_g - u_p) [\frac{1}{\nu} (\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y})]^{1/2} \qquad (1.23)$$

$$T = \frac{\alpha \rho_{sp}}{\tau_r} \gamma_j^2 [\frac{1}{2} (\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}) - \omega] \qquad (1.24)$$

In eqs. (1.22) to (1.24), $(u_g, v_g)$ and $(u_p, v_p)$ respectively denote the components of velocities of gas and particle phases for the two dimensional flow in $x$ and $y$ directions, $\omega$ the angular velocity of the particles, $\gamma_j$ the radius of gyration of a spherical solid particle about an axis passing through its center, $\nu$ the kinematic viscosity of the fluid and

$$\tau_T = \frac{d^2 \rho_{sp}}{18\ \mu_g}$$

$$\tau_r = \frac{d^2 \rho_{sp}}{60\ \mu_g}$$

Here, $\tau_T$ and $\tau_r$ are called relaxation times for particle translation and rotation respectively.

Using this formula for resistance, they solved the gas particulate flow due to the impulsive motion of an infinite plate and found that a demixed region with no particles present developed near the plate. This is due to the particles migrating away from the wall.

$g \alpha \rho_{sg} \delta_{i\ell}$ is the buoyancy force acting on the particles.

Energy equations

Hinze derived the equations conserving the mechanical energy to determine pressure. He did not derive the equation conserving thermal energy and the equation of state. Probably, for these reasons, Murray (1965) has stated that Hinze (1963) did not consider gas phase as compressible. Murray (1965) derived the equation representing the conservation of mechanical and thermal energy of the mixture and then using the momentum equations, got the energy equations for separate phases. Drew and Seigal (1963) deduced the thermal energy equation consistent with their approach of using average quantities for the various variables involved in the problem. Pai (1970) got the energy equations for separate phases but did not consider certain interaction forces. In the present analysis, we have derived the energy equations for separate species, taking care of the interaction forces as represented by virtual mass, buoyancy and pressure gradient besides the resistance force. This derivation is consistent with our approach adopted in deriving the momentum equations. We have also put these equations in terms of specific enthalpy for the gas phase and specific energy for the particulate phase. These forms are for ready use in solving specific problem involving compressible gas phase.

The corresponding energy equations for the gas and particle phases can be written as

$$\frac{D}{Dt_g}\left[\rho_g\left(e_g+\frac{1}{2}u_{gi}^2\right)\right] = \frac{\partial}{\partial x_j}\left[u_{gi}(-\quad\delta_{ij}+\tau_{ij})+q_{gj}\right]$$

$$-(1-\alpha)\ \rho_{sg}\ g\ \delta_{i\ell}\ u_{gi} + u_{pi}\ F_i + Q_p \qquad (1.25)$$

$$\frac{D}{Dt_p}\left[\rho_p\left(e_p+\frac{1}{2}u_{pi}^2\right)\right] = -u_{pi}\ \alpha\rho_{sp}\ g\delta_{i\ell} - u_{pi}F_i - Q_p \qquad (1.26)$$

Here, $e_g$ and $e_p$ denote respectively the specific energies for gas and particle phases, $q_{gj}$ represents the heat flux across the fluid element and is given by

$$q_{gj} = -k\frac{\partial T_g}{\partial x_j}$$

k being the coefficient of thermal conductivity of the fluid.

In eq. (1.25), $Q_p$ is the heat transferred from the particle cloud to the gas phase. The particles may have different temperature from the surrounding fluid. Due to this temperature difference, there will be heat transfer between the two phases. For the Stokes flow, heat transferred from the particles to the gas is given by

$$Q_p = \frac{\rho_p C_s(T_p - T_g)}{T_t}$$

where

$$T_t = \frac{mC_p}{4\pi ak}$$

is the relaxation time for temperature, $C_s$ is the specific heat of solid particles and $C_p$ is the specific heat of fluid at constant pressure.

Multiply the momentum equation (1.8) by $u_{gi}$ and then subtract the resulting equation from eq. (1.25). Using the continuity equation (1.6), we get the equation of energy for the fluid phase as

$$\rho_g \frac{De_g}{Dt_g} = -p \frac{\partial u_{gi}}{\partial x_i} + \tau_{ij} \frac{\partial u_{gi}}{\partial x_j} + (Q_p + \frac{\partial q_{gj}}{\partial x_j})$$

$$+(u_{pi}-u_{gi})\, F_i \qquad (1.27)$$

$$e_g = I_g - \frac{p}{\rho_g} = C_p T_g - \frac{p}{\rho_g} \qquad (1.28)$$

$I_g$ being the specific enthalpy of the fluid phase. Assuming $C_p$ for the fluid phase as constant, eq. (1.28) after differentiation and then using continuity equation (1.6) gives

$$\rho_g \frac{De_g}{Dt_g} = \rho_g C_p \frac{DT_g}{Dt_g} - \frac{Dp}{Dt_g} - p \frac{\partial u_{gi}}{\partial x_i} \qquad (1.29)$$

with eq. (1.29), eq. (1.27) becomes

$$\rho_g C_p \frac{DT_g}{Dt_g} = \frac{Dp}{Dt_g} + \tau_{ij} \frac{\partial u_{gi}}{\partial x_j} + (Q_p + \frac{\partial q_{gj}}{\partial x_j})$$

$$+ (u_{pi} - u_{gi})\, F_i \qquad (1.30)$$

Similarly multiply equation (1.9) by $u_{pi}$ and subtract the resulting equation from (1.26),using the continuity equation (1.7), we get the equation of energy for the particle phase as

$$\rho_p \frac{De_p}{Dt_p} = - Q_p \qquad (1.31)$$

Also

$$e_p = C_s T_p \qquad (1.32)$$

with eq. (1.32), eq. (1.31) becomes

$$\rho_p \ C_s \ \frac{DT_p}{Dt_p} = - Q_p \qquad (1.33)$$

Eqs. (1.6) to (1.9), (1.18), (1.30) and (1.33) form a set of seven equations for seven unknowns viz. $u_{gi}$, $u_{pi}$, $\alpha$, $\rho_g$, p, $T_g$ and $T_p$. For a particular flow problem, the above equations are solved with suitable boundary conditions.

With rotating particles in the fluid, we need consider an additional equation representing the conservation of angular momentum of the particulate phase. Hamed and Tabakoff (1974) gave the following equation for the conservation of angular momentum of the solid particles.

$$\rho_p \ \frac{D\omega}{Dt_p} = T/\gamma_j^2$$

where 'T' is the torque of bulk interaction per unit volume between the particle and the fluid.

Hamed and Tabakoff (1975) further found that the presence of the solid particles with rotation causes an additional anti-symmetric stress tensor and gave the following expressions:

$$\sigma_{xx} = -p - \frac{2}{3} \ \mu_g\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu_g \frac{\partial u}{\partial x}$$

$$\sigma_{xy} = \mu_g\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\alpha}{2} \ \rho_p \ \gamma_j^2 \ \frac{D\omega}{Dt_p}$$

$$\sigma_{yy} = -p - \frac{2}{2} \ \mu_g\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu_g \frac{\partial u}{\partial y}$$

$$\sigma_{yx} = \mu_g\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\alpha}{2} \ \rho_p \ \gamma_j^2 \ \frac{D\omega}{Dt_p}$$

where $\sigma_{xx}$, $\sigma_{xy}$, $\sigma_{yx}$ and $\sigma_{yy}$ are the elements of the fluid

stress tensor, (u,v) the components of velocity of the fluid phase in a two dimensional suspension flow.

Particular form of the governing equations of motion

We simplify eqs. (1.6) to (1.9), (1.18), (1.30) and (1.33) by making the following assumptions:

(i) Gas phase is incompressible i.e. $\rho_g$ is constant but the particle phase is compressible.

(ii) The temperature difference between the fluid and the particles is negligible.

(iii) Volume fraction $\alpha$ of the particles is negligible.

(iv) Body forces, rotation of the particles and the lift forces are negligible.

(v) The particles are sparsely distributed and spherical in shape.

With these assumptions eqs. (1.14) and (1.15) give

$$\mu_m = \mu_g = \mu \text{ (say)} \tag{1.38}$$

and the viscous stress tensor $\tau_{ij}$ (eq. (1.13)) becomes

$$\tau_{ij} = \mu\left(\frac{\partial u_{gi}}{\partial x_j} + \frac{\partial u_{gj}}{\partial x_i}\right) - \frac{2}{3}\mu \frac{\partial u_{gk}}{\partial x_k}\delta_{ij} \tag{1.37}$$

The equations of continuity and momentum for the two phases become

$$\frac{\partial}{\partial x_i}(u_{gi}) = 0 \tag{1.40}$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_i}(\rho_p u_{pi}) = 0 \tag{1.41}$$

$$\rho_g \frac{Du_{gi}}{Dt_g} = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} (\tau_{ij}) - F\rho_p(u_{gi}-u_{pi}) \tag{1.42}$$

$$\rho_p \frac{Du_{pi}}{Dt_p} = F\rho_p(u_{gi}-u_{pi}) \tag{1.43}$$

Energy equations (1.30) and (1.33) become redundant in view of the assumptions (i) and (ii).

Here, the unknowns are $u_{gi}$, $p$ , $\rho_p$ and $u_{pi}$. For a 2-D flow, the number of unknowns is six and we have six equations to determine them. Hence, with prescribed boundary conditions, it should be possible to solve a gas-particulate flow problem.

Further, we notice that with the above assumptions, the gas phase realizes the presence of the particles only through the resistance term. Similarly, the particle phase equations are linked with the gas phase equations through the resistance term. For a particular flow problem, we have to solve the above set of equations simultaneously.

## 1.3 Laminar boundary layer of gas particulate flows

The concept of viscous flow boundary layer can be extended to gas-particulate flows. The boundary layers with particle suspension may be turbulent. In the present study, laminar boundary layer of gas particle suspension is studied with the purpose of developing a basic understanding of the interaction of gas-particle flow near the solid wall.

The study of the boundary layer of gas-particulate flow is important for the following reasons:

(i) To find the effect of the presence of the particles on the structure of the boundary layer and its surface characteristics such as skin friction.

(ii) To study the particle retardation and accumulation on the surface.

Here, we shall derive the boundary layer equations of the gas-particulate flow by an order of magnitude analysis for the incompressible gas phase and compressible particle phase. For a 2-D steady flow, the basic governing equations (1.40) to (1.43) are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.44}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - F\rho_p(u-u_p) \tag{1.45}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - F\rho_p(v-v_p) \tag{1.46}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 \tag{1.47}$$

$$\rho_p\left(u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right) = F\rho_p(u-u_p) \tag{1.48}$$

$$\rho_p\left(u_p\frac{\partial v_p}{\partial x} + v_p\frac{\partial v_p}{\partial y}\right) = F\rho_p(v-v_p) \tag{1.49}$$

Here the suffix 'g' has been dropped from the variables for the gas phase.

Based on the Prandtle's assumption, the effect of viscosity near a solid wall is realized only upto a small distance '$\delta$'. '$\delta$' is small in comparison to other dimensions involved in the flow problem. If $x \sim O(1)$, then $y \sim O(\delta)$ and the continuity equation (1.44) gives that $v \sim O(\delta)$. From the momentum equation (1.45), we derive that $\mu$ must be of $O(\delta^2)$ in order to retain the same order of inertia and viscous terms. From eq. (1.46) it can be easily shown that $(v_p - v)$ is of the same order as the inertia terms. Since for the gas flow, inertia terms are of order $\delta$, it gives $(v_p - v) = O(\delta)$. Hence $O(v_p) = O(v)$ $= O(\delta)$. Thus the boundary layer equations for fluid phase are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.50}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - F\rho_p (u - u_p) \tag{1.51}$$

$$0 = - \frac{\partial p}{\partial y} \tag{1.52}$$

Following the same reasoning, the corresponding equations for the particulate phase are the following:

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v_p) = 0 \tag{1.53}$$

$$\rho_p u_p \frac{\partial u_p}{\partial x} + \rho_p v_p \frac{\partial u_p}{\partial y} = F\rho_p (u - u_p) \tag{1.54}$$

$$\rho_p u_p \frac{\partial v_p}{\partial x} + \rho_p v_p \frac{\partial v_p}{\partial y} = F \rho_p (v - v_p) \tag{1.55}$$

The eqs. (1.53) to (1.55) are the same in form as equations (1.47) to (1.49). With boundary layer concept, eq. (1.55) is of the order '$\delta$' while eqs. (1.53) and (1.54) are of order unity. With compressible particle phase assumptions, the unknowns for the particle phase are three viz. $\rho_p, u_p$ and $v_p$. Hence, we need three equations to determine them. Thus, for mathematical consistency, we have to retain the equation (1.55) in the system of equations to describe the boundary layer flow. It is presumed here that retaining of lower order equation viz. eq. (1.55) will not effect the over all accuracy of the system of equations. It may probably improve upon it.

Marble (1962), Soo (1967), Tabakoff and Hamed (1972) neglected eq. (1.55) in their formulation of the problem. They assumed $v_p = v$, then effectively reducing the number of unknowns by unity. Soo (1968), Singleten (1966) and Otterman (1969) retained the equation (1.55) but kept $v_p \neq v$.

## 1.4 Boundary conditions

For gas-particulate boundary layer flows, the boundary conditions for the gas phase are the following:

At $\quad y = 0, \quad u = v = 0 \qquad (1.56)$

As $\quad y \to \infty, \quad u = u_\infty \qquad (1.57)$

where $u_\infty$ is the free stream velocity at the edge of the boundary layer.

For particle phase, the boundary conditions are:

$$\text{At} \quad y = 0, \quad v_p = 0 \tag{1.58}$$

$$\text{As} \quad y \to \infty, \; u_p = u_{p\infty} \text{ and } \rho_p = \rho_{p\infty} \tag{1.59}$$

where $u_{p\infty}$ and $\rho_{p\infty}$ are the velocity and density of the pseudo fluid of particles at the edge of the boundary layer.

The particle phase being pseudo fluid lacks definite boundary condition on $u_p$ on the surface. This condition is provided by evaluating the continuity and momentum equations for the particle phase at the surface. The analytical conditions so obtained are known as compatibility conditions. Eq. (1.54), when evaluated at the wall gives

$$\frac{\partial u_p}{\partial x}\Big|_{y=0} = -F \quad \text{or} \quad u_{pw} = u_{p\infty} - Fx \tag{1.60}$$

The continuity equation (1.50) at the wall gives

$$\frac{\partial v}{\partial y}\Big|_{y=0} = 0 \quad (u = 0 \quad \text{at } y = 0 \text{ for all } x)$$

Since $v_p = O(v)$, we can take $\frac{\partial v_p}{\partial y}\Big|_{y=0} = 0$

$\therefore$ eq. (1.53), evaluated at the wall gives

$$\frac{\partial}{\partial x}(\rho_p u_p)\Big|_{y=0} = 0 \quad \text{or} \quad \frac{\rho_{pw}}{\rho_{p\infty}} = \frac{1}{1-\frac{Fx}{u_{p\infty}}} \tag{1.61}$$

Conditions (1.60) and (1.61) are termed as compatibility conditions for the particle phase.

Compatibility conditions for the gas phase only were first used by Pohlhauson (1921) while solving Karman integral of

boundary layer equations. Over a period of time, the use of these conditions in integral methods have given results that are resonably in accord with the exact solution of the boundary layer equations.

Higher order compatibility conditions can also be obtained by differentiating momentum equation and evaluating the resulting equation at the wall [Schlichting (1962)]. Jain and Bhatnagar (1962) used these higher order compatibility conditions for boundary layer flows with pressure gradient and got reasonably accurate results.

Researchers justify the use of these conditions in solving boundary layer equations by the various arguments. A few of them are listed below:

(i) Since in the boundary layer flow, we are concerned greatly with the condition near the wall, more boundary conditions should be used near the wall rather than at the edge of the boundary layer.

(ii) Boundary layer equations are valid at every point inside the boundary layer. Hence, these equations when evaluated at the wall should provide an additional boundary condition.

In gas-particulate flow problem, Soo (1967), Tabakoff and Hamed (1972) used the compatibility conditions (1.60) and (1.61) for the particle phase in the development of the

integral method. Marble (1962) in developing the series solution did not use the compatibility conditions (1.60) and (1.61). The results of these two approaches to solve the same problem differ significantly both qualitatively and quantitatively. For example, from Soo (1967) and Tabakoff's and Hamed analysis (1972), we find that surface particulate velocity varies linearly with the longitudinal distance $x^* = (\frac{xF}{u_\infty})$ and at $x^*=1$, $u_{pw} = 0$ and $\rho_{pw}$ tends to infinity. Marble's analysis gives $u_{pw} = 0$ for all x upto the first order of the series solution. This differing nature of the results gave rise to the question of validity of the compatibility conditions for the particle phase. Also, one finds that imposition of the condition $u_{pw} = 1-x^*$ in the analysis forces the solution to give the result that $u_{pw} \propto 1-x^*$. In view of these observations, it was found necessary to probe further the use of the compatibility conditions in the analysis of the boundary layer flow.

## 1.5 Author's contribution

In chapter II, gas-particulate boundary layer equations (1.50) to (1.55) are integrated numerically with and without the use of compatability conditions. We use the finite difference scheme due to Crank-Nicholson to integrate the governing equations (1.50) to (1.54) with compatibility conditions and equations (1.50) to (1.55) without compatibility

conditions. It is interesting to note that both the analyses give linear variation of the surface particulate velocity. The analysis without compatibility conditions predicts better structure of the flow.

In chapter III, with a view to understand some basic characteristics of the gas-particulate flow, we obtain some exact solutions of the eqs. (1.40) to (1.43) in cartesian coordinates. Problems considered are:

(i) the steady flow between infinite porous parallel plates when the lower plate is at rest and the upper plate moves with uniform velocity. The lower plate is subjected to a constant suction (injection) velocity while the upper plate is subjected to injection (suction) at the same rate

(ii) unsteady flow between infinite porous parallel plates when the upper plate starts moving with the velocity increasing or decreasing exponentially with time. At time $t = 0$, we get the steady flow case as a particular case. It is observed that velocity of either phase increases as the suction velocity at the lower plate increases. In case, the upper plate moves with velocity increasing (decreasing) with time, skin friction at the lower plate increases (decreases) non linearly with time. It is further observed that when the upper plate moves with accelerating velocity, velocity profiles for the gas and particle phases are almost linear and the difference between the velocities of both the phases increases with time. In

case, the upper plate moves with velocity decreasing with time, the difference between the velocities of both the phases decreases with time.

In chapter IV, we solve Navier-Stokes equations (1.40) to (1.43) with incompressible fluid phase and compressible particle phase in cylindrical polar coordinates. The physical problem solved is the gas-particulate flow in tubes of varying cross-section. Blasius (1910), Tanner (1966), Lee and Fung (1970) and Manton (1971) solved the same problem for fluid phase only. Kaimal (1977) extended the work of Manton for gas-particulate flows. He considered both the gas phase and particle phase as incompressible. He found analytical expressions for the first three terms of the series solution. His solution indicates the formation of eddies even at Reynolds number of unity.

In the present investigation, we modify Kaimal's analysis to compressible particle phase. We develop an asymptotic series in terms of the small parameter $\epsilon$, which gives the rate of variation of the cross-section of the tube with the axial distance and calculate first three terms. It is obtained that particles move radially towards the wall which results in excessive accumulation of the particles there. We further observe that the shear stress at the wall decreases by adding the coarse dust particles. The present analysis is strictly valid for low Reynolds number. In case, we compute the

results for high Reynolds numbers, eddies are formed at the wall and the separation of gas and particulate phases take place.

In chapter V, we solve eqs. (1.40) to (1.43) in cylindrical polar coordinates and solve the problem of gas-particulate flow in the curved pipe. Here, we assume that radius of the circle in which central line of the pipe is coiled is large in comparison to the radius of cross-section of the pipe. Also, the secondary motion for the gas phase is replaced by the uniform stream. Dean (1959) solved the problem for fluid phase in the curved pipe and found that the volume flow rate decreases as the secondary flow increases through the pipe. In the present investigation, secondary motion for the particulate phase is calculated from the governing equations. We use the method of Green's function to solve the boundary value problem so obtained. It is observed that in case the particles are fine, the volume flow rate for the clean gas is more than the volume flow rate for gas-particulate flow. In the case of coarse particles, gas-particulate flow predicts more volume flow rate than the one predicted by clean gas. In both the cases it is obtained that the region where the primary motion is greatest, shifts outward.

In chapter VI, we discuss the stability and uniqueness criterion of an initial boundary value problem of a dusty gas model described by eqs. (1.40) to (1.43). There are certain investigations in this direction. Crooke (1976) examined a

uniqueness criterion for a dusty gas model and Dandapat and Gupta (1976) obtained a universal stability theorem for a dusty gas model. In both the investigations, authors considered constant number density of the particle phase. Dandapat and Gupta (1976) did not obtain any uniqueness criterion while Crooke (1976) did not bother about universal stability theorem.

In the present investigation, we improve the universal stability theorem of Dandapat and Gupta (1976) and as a consequence, we establish a uniqueness criterion. At one point, we depart from the analysis of Dandapat and Gupta to obtain a much improved theorem on universal stability. It ensures a wider range of stability region. We further generalize the analysis to cover the case of non uniform number density of particle phase. We get only the uniqueness theorem in this case.

# CHAPTER II

## GAS-PARTICULATE BOUNDARY LAYER FLOW ON A FLAT PLATE*

### 2.1 Introduction

In the present chapter, the problem of boundary layer for gas-particulate flow has been solved with a view to understand the structure of the boundary layer flow and the use of the various boundary conditions. This chapter consists of two parts. In part I, we solve the gas-particulate boundary layer equations by finite difference scheme due to Crank-Nicholson using compatibility conditions for the particle phase. In part II, the gas-particulate boundary layer equations are integrated numerically with and without the use of compatibility conditions.

Part I

The solution of the governing equations of gas-particulate boundary layer flow has been obtained by the series method and by the approximate methods. The results by the two approaches differ in several essential aspects. Marble (1962) developed the series solution for an incompressible gas phase and compressible particle phase and solved the first two terms of the series expansion. His analysis indicated that the

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* The contents of this chapter have been published in Refs. [45,46].

particle velocity on the surface remained zero upto the first order term of the series solution. Soo (1967) solved the same problem by momentum integral method using linear profiles for the gas and for the particulate velocities. Takakoff and Hamed (1972) modified the procedure by taking a fourth degree profile for the gas phase and a similar profile for the particle phase. These analyses showed that the surface particulate velocity decreased linearly with the longitudinal distance 'x'. Also, particulate velocity in the boundary layer decreased near the surface and then increased monotonically to attain the free stream value at the edge of the boundary layer. For some values of the governing parameters, it became negative inside the boundary layer. These features of the gas-particulate boundary layer seemed unplausible. Hence, a need was felt to find exact solution of these equations.

We solve the same problem by integrating the governing equations of the two phase boundary layers numerically by a finite difference scheme due to Crank-Nicholson. We find that the present procedure predicts a better structure of the flow than is given by approximate method of Tabakoff and Hamed (1972). Here, the particulate velocity remains positive and increases monotonically from the surface value to its free stream value. Also, the present results fail at a much later station on the flat plate than the results of Tabakoff and

Hamed (1972). Graphs are drawn to illustrate the effect of the changes in non-dimensional parameter $\bar{F}$ on the coefficient of skin-friction $C_f$ and the particulate velocity $u_p$.

Here, the particles are assumed to be spherical in shape. They are supposed to be sparsely distributed so that a particle is not submerged in the wake of any other particle. For the mathematical simplicity, Stokes drag law is used.

## 2.2 Governing equations and boundary conditions

The governing equations for a steady 2-dimensional gas-particulate boundary layer flow on a flat plate are derived in chapter I [eqs. (1.50) to (1.55)]. We further assume that $v_p = v$. As such, these equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} - F \rho_p (u - u_p) \tag{2.2}$$

$$\frac{\partial}{\partial x}(\rho_p u_p) + \frac{\partial}{\partial y}(\rho_p v) = 0 \tag{2.3}$$

$$u_p \frac{\partial u_p}{\partial x} + v \frac{\partial u_p}{\partial y} = F(u - u_p) \tag{2.4}$$

Here, x,y are the coordinates measured along and perpendicular to the plate length, u and $u_p$ the longitudinal velocity components for the gas and particle phases respectively and v the transverse component of the velocity for both

the phases. $\rho$ and $\rho_p$ are the densities of the gas and particle phases respectively, $\mu$ the coefficient of viscosity for the gas phase and

$$F = \frac{18\,\mu}{d^2\,\rho_{sp}}$$

d being the diameter of a spherical particle and $\rho_{sp}$ the material density of the particles.

Boundary conditions

I For gas phase

(i) At $y = 0,\ u = v = 0$ (2.5)

(ii) As $y \to \infty,\ u = u_\infty$ (2.6)

II For particle phase

Since $u = 0$ at $y = 0$ for all x, eq. (2.1) when evaluated at the wall gives

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} = 0 \qquad (2.7)$$

Evaluating eq. (2.3) at the wall and then using eqs. (2.5) and (2.7), we get

(i) $$\left.\frac{\partial}{\partial x}(\rho_p u_p)\right|_{y=0} = 0 \qquad (2.8)$$ Compatibility

Evaluating eq. (2.4) at the wall and then using conditions given in (2.5), we get

(ii) $$\left.\frac{\partial u_p}{\partial x}\right|_{y=0} = -F \qquad (2.9)$$ Compatability

Conditions given in eqs. (2.8) and (2.9) are termed as compatibility conditions for the particle phase.

(iii) As $y \to \infty$, $u_p = u_\infty$ and $\rho_p = \rho_{p\infty}$ (2.10)

Here, $u_\infty$ is the velocity of both the phases in the free stream and $\rho_{p\infty}$ the density of the particles in the free stream.

We take the initial profiles from the approximate results of Tabakoff and Hamed (1972) and Jain and Ghosh (1979).

We define the non-dimensional variables as follows:

$$\left.\begin{array}{l} \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y\sqrt{R_e}}{L}, \quad \bar{u} = \frac{u}{u_\infty}, \quad \bar{u}_p = \frac{u_p}{u_\infty} \\ \bar{v} = \frac{v\sqrt{R_e}}{u_\infty}, \quad \bar{\rho}_p = \frac{\rho_p}{\rho_{p\infty}}, \quad D = \frac{d}{L}, \quad \bar{\rho}_{sp} = \frac{\rho_{sp}}{\rho_{p\infty}} \\ \text{and} \qquad \bar{\rho} = \frac{\rho}{\rho_{p\infty}} \end{array}\right] \tag{2.11}$$

where $L$ is the characteristic length and

$$R_e = \frac{\rho u_\infty L}{\mu}$$

Using (2.11), eqs. (2.1) to (2.4) in the nondimensional form become

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}} = 0 \tag{2.12}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F}\,\bar{\rho}_p\,(\bar{u}-\bar{u}_p) \tag{2.13}$$

$$\frac{\partial}{\partial \bar{x}}(\bar{\rho}_p \bar{u}_p) + \frac{\partial}{\partial \bar{y}}(\bar{\rho}_p \bar{v}) = 0 \tag{2.14}$$

$$\bar{u}_p\frac{\partial \bar{u}_p}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}_p}{\partial \bar{y}} = \bar{F}\,\bar{\rho}\,(\bar{u}-\bar{u}_p) \tag{2.15}$$

where

$$\bar{F} = \frac{18}{R_e D^2 \bar{\rho}_{sp}}$$

Boundary conditions become:

I For gas phase

(i) At $y = 0$, $\bar{u} = \bar{v} = 0$ (2.16)

(ii) As $\bar{y} \to \infty$, $\bar{u} = 1$ (2.17)

II For particle phase

(i) $$\frac{\partial}{\partial \bar{x}} (\bar{\rho}_p \bar{u}_p) \Big|_{\bar{y}=0} = 0 \qquad (2.18)$$

(ii) $$\left[\frac{\partial \bar{u}_p}{\partial \bar{x}}\right]_{\bar{y}=0} = - \bar{F}\ \bar{\rho} \qquad (2.19)$$

(iii) As $\bar{y} \to \infty$, $\bar{u}_p = \bar{\rho}_p = 1$ (2.20)

Initial profiles

I Initial profiles for $\bar{u}, \bar{v}, \bar{u}_p$ and $\bar{\rho}_p$ corresponding to the approximate results of Tabakoff and Hamed (1972) are the following:

$$\bar{u} = 2\eta - 2\eta^3 + \eta^4 + \frac{\Lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4) \qquad (2.21)$$

$$\bar{v} = \eta^2 - \frac{1}{6}\eta^3 - \frac{5}{4}\eta^4 + \frac{7}{10}\eta^5 - \frac{\Lambda}{120}(10\eta^2 - 20\eta^3 + 15\eta^4 - 4\eta^5)\frac{\delta}{2\Lambda}\frac{d\Lambda}{dx} \qquad (2.22)$$

$$\bar{u}_p = 1 - \frac{Fx}{u_\infty}\left[1 - 2\eta_p + 2\eta_p^3 - \eta_p^4 + \left(6 - \frac{8}{1-0.6\frac{Fx}{u_\infty}}\right)(\eta_p^4 - 3\eta_p^3 + 3\eta_p^2 - \eta_p)\right] \qquad (2.23)$$

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$$\bar{\rho}_p = 1 + \frac{Fx|u_\infty}{1-Fx|u_\infty}\,(1-6\eta^2 + 8\eta^3 - 3\eta^4) \qquad (2.24)$$

where

$$\eta = \frac{y}{\delta}\,, \quad \eta_p = \frac{1}{\delta_{\rho_p}} \int \bar{\rho}_p \; dy, \quad = \frac{F\delta^2 \rho_{p\infty}}{\mu}$$

and

$$z = \frac{Fx}{u_\infty}$$

$\delta$ and $\delta_{\rho_p}$ are calculated from the following equations

$$\frac{d\delta}{dz}\left(\frac{37}{315} - \frac{\Lambda}{315} - \frac{5\Lambda^2}{9072}\right) = \frac{\mu}{F\rho\delta}\left(2 + \frac{\Lambda}{6}\right)$$

$$- \frac{\rho_{p\infty}}{\rho}\,\delta\,\left[\frac{3}{10} + \frac{19}{84}\,\frac{z}{1-z} - \frac{\Lambda}{6}\left(\frac{1}{20} + \frac{5}{168}\,\frac{z}{1-z}\right)\right]$$

$$+ \frac{\rho_{p\infty}}{\rho}\,\delta_{\rho_p}\,\frac{0.4z}{1-0.6z} \qquad (2.25)$$

and

$$\left[0.3z - \frac{23}{126}\,z^2 - \left(0.3 - \frac{0.4}{1-0.6z}\right)\left(z - \frac{110}{126}\,z^2\right)\right.$$

$$\left. - \frac{100}{63}\,z^2\left(0.3 - \frac{0.4}{1-0.6z}\right)^2\right]\frac{d\delta_{\rho_p}}{dz}$$

$$= \left[\frac{3}{10} + \frac{19}{84}\,\frac{z}{1-z} - \frac{\rho_{p\infty}F\delta^2}{6\mu}\left(\frac{1}{20} + \frac{5}{168}\,\frac{z}{1-z}\right)\right]\delta$$

$$- \left[\frac{0.4z}{1-0.6z} + 0.3 - \frac{23}{63}\,z - \left(0.3 - \frac{0.4}{1-0.6z}\right)\right.$$

$$\left(1 - \frac{110}{63}\,z\right) + \frac{0.24}{(1-0.6z)^2}\left(z - \frac{110}{126}\,z^2\right)$$

$$- \frac{200}{63}\,z\left(0.3 - \frac{0.4}{1-0.6z}\right)^2 + \frac{200}{63}\,z^2\left(0.3 - \frac{0.4}{1-0.6z}\right)$$

$$\left.\frac{0.24}{(1-0.6z)^2}\right]\delta_{\rho_p} \qquad (2.26)$$

The first term on L.H.S. of eq. (2.25) is different than that of the corresponding term in eq. (44) of Tabakoff and Hamed's (1972) paper, because of miscalculation there.

We integrate eqs. (2.25) and (2.26) by Taylor's series method under the boundary conditions $\delta = \delta_{\rho_p} = 0$ at $z = 0$ and calculate the profiles for $\bar{u}$, $\bar{v}$, $\bar{u}_p$ and $\bar{\rho}_p$ at $\bar{x} = 0.1$ from eqs. (2.21) to (2.24).

II Initial profiles corresponding to the approximate solution of Jain and Ghosh (1979) are

$$\bar{u} = 1-(1-\eta)^3 \tag{2.27}$$

$$\bar{v} = \left(\frac{3}{2}\eta^2 - 2\eta^3 + \frac{3}{4}\eta^4\right)\frac{d\delta}{dx} \tag{2.28}$$

$$\bar{u}_p = 1-(1-a_2)(1-\eta)^3 \tag{2.29}$$

$$\bar{\rho}_p = 1-(1-a_3)(1-\eta)^3 \tag{2.30}$$

where

$$\eta = \frac{y}{\delta},\ a_2 = \bar{u}_{pw},\ a_3 = \bar{\rho}_{pw},\ A = \frac{\delta^2}{L^2}.$$

$u_{pw}$ and $\rho_{pw}$ being the velocity and density of the particle phase at the wall.

A, $a_2$ and $a_3$ are calculated from the following equations

$$\frac{3}{56}\frac{dA}{d\bar{x}} = \frac{3\mu}{\rho u_\infty L} - \frac{FL}{u_\infty}\frac{\rho_{p\infty}}{\rho} A\left(\frac{5}{14} - \frac{3}{28}a_3\right) \tag{2.31}$$

$$\frac{1}{2}\frac{dA}{d\bar{x}}\left(\frac{221}{560} - \frac{83}{224}a_2 - \frac{9}{160}a_3 + \frac{9}{280}a_2^2\right) + \frac{36}{35}A\frac{da_2}{d\bar{x}}$$
$$(a_2+.072)(a_2-5.696)(a_2+1.364)/(4a_2+3)^2 = \frac{FL}{u_\infty}.A\left(\frac{5}{14} - \frac{3}{28}a_3\right) \tag{2.32}$$

$$a_2 a_3 = \frac{5}{2} - \frac{3}{4}(a_2 + a_3) \qquad (2.33)$$

We integrate eqs. (2.31) to (2.33) by Tayler's series expansion under the boundary conditions

At $x = 0$, $A = 0$, $a_2 = a_3 = 1$

and calculate the profiles for $\bar{u}, \bar{v}, \bar{u}_p$ and $\bar{\rho}_p$ at $\bar{x} = 0.1$

## 2.3 Method of solution

We integrate eqs. (2.12) to (2.15) together with the boundary conditions (2.16) to (2.20) by a finite difference scheme due to Crank-Nicholson. The various terms in eqs. (2.12) to (2.15) are replaced by the following differences:

$$\bar{u}_{m+1/2,n} = \frac{1}{2}(\bar{u}_{m+1,n} + \bar{u}_{m,n}) \qquad (2.34)$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_{m+1/2,n} = \frac{1}{\Delta \bar{x}}(\bar{u}_{m+1,n} - \bar{u}_{m,n}) \qquad (2.35)$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{m+1/2,n} = \frac{1}{4\Delta \bar{y}}(\bar{u}_{m+1,n+1} - \bar{u}_{m+1,n-1} + \bar{u}_{m,n+1} - \bar{u}_{m,n-1}) \qquad (2.36)$$

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{m+1/2,n} = \frac{1}{2(\Delta \bar{y})^2}(\bar{u}_{m+1,n+1} - 2\bar{u}_{m+1,n} + \bar{u}_{m+1,n-1} + \bar{u}_{m,n+1} - 2\bar{u}_{m,n} + \bar{u}_{m,n-1}) \qquad (2.37)$$

Here, the suffixes m,n represent the value of the variable at the grid point represented by $m^{th}$ step in x-direction and $n^{th}$ step in y-direction.

Similar expressions for $\bar{v}$, $\bar{u}_p$, $\bar{\rho}_p$ and their derivatives are used in eqs. (2.12) to (2.15). We difference eq. (2.12) at the point $(m+\frac{1}{2}, n-\frac{1}{2})$ and eqs. (2.13) to (2.15) at the

point $(m + \frac{1}{2}, n)$. The resulting difference equations are:

$$\bar{v}_{m+1/2,n} = \bar{v}_{m+1/2,n-1} - \frac{\Delta\bar{y}}{2\Delta\bar{x}}(\bar{u}_{m+1,n} + \bar{u}_{m+1,n-1} - \bar{u}_{m,n} - \bar{u}_{m,n-1}) \qquad (2.38)$$

$$A_n \bar{u}_{m+1,n+1} + B_n \bar{u}_{m+1,n} + C_n \bar{u}_{m+1,n-1} = D_n \qquad (2.39)$$

where

$$A_n = \frac{\bar{v}_{m+1/2,n}}{4\Delta\bar{y}} - \frac{1}{2(\Delta\bar{y})^2}$$

$$B_n = \frac{\bar{u}_{m+1,n}}{\Delta\bar{x}} + \frac{1}{(\Delta\bar{y})^2} + \frac{\bar{F}}{2}(\bar{\rho}_{p_{m+1,n}} + \bar{\rho}_{p_{m,n}})$$

$$C_n = -\frac{\bar{v}_{m+1/2,n}}{4\Delta\bar{y}} - \frac{1}{2(\Delta\bar{y})^2}$$

$$D_n = \frac{1}{2\Delta\bar{x}}(\bar{u}^2_{m+1,n} + \bar{u}^2_{m,n}) - \frac{\bar{v}_{m+1/2,n}}{4\Delta\bar{y}}(\bar{u}_{m,n+1} - \bar{u}_{m,n-1}) + \frac{1}{2(\Delta\bar{y})^2}(\bar{u}_{m,n+1} - 2\bar{u}_{m,n} + \bar{u}_{m,n-1}) - \frac{\bar{F}}{4}(\bar{\rho}_{p_{m+1,n}} + \bar{\rho}_{p_{m,n}})(\bar{u}_{m,n} - \bar{u}_{p_{m+1,n}} - \bar{u}_{p_{m,n}})$$

$$A^*_n \bar{\rho}_{p_{m+1,n+1}} + B^*_n \bar{\rho}_{p_{m+1,n}} + C^*_n \bar{\rho}_{p_{m+1,n-1}} = D^*_n \qquad (2.40)$$

where

$$A^*_n = \frac{\bar{v}_{m+1/2,n}}{4\Delta\bar{y}}$$

$$B^*_n = \frac{\bar{u}_{p_{m+1,n}}}{\Delta\bar{x}} + \frac{1}{4\Delta\bar{y}}(\bar{v}_{m+1/2,n} - \bar{v}_{m+1/2,n-1})$$

$$C_n^* = -A_n^*$$

$$D_n^* = \frac{\bar{u}_{p_{m,n}} \, \bar{\rho}_{p_{m,n}}}{\Delta \bar{x}} - \frac{\bar{\rho}_{p_{m,n}}}{4 \Delta \bar{y}} (\bar{v}_{m+1/2,n} - \bar{v}_{m+1/2,n-1})$$

$$- \frac{\bar{v}_{m+1/2,n}}{4 \Delta \bar{y}} (\bar{\rho}_{p_{m,n+1}} - \bar{\rho}_{p_{m,n-1}})$$

$$A_n^{**} \, \bar{u}_{p_{m+1,n+1}} + B_n^{**} \, \bar{u}_{p_{m+1,n}} + C_n^{**} \, \bar{u}_{p_{m+1,n-1}} = D_n^{**} \quad (2.41)$$

where

$$A_n^{**} = \bar{v}_{m+1/2,n} / (4 \Delta \bar{y})$$

$$B_n^{**} = \frac{\bar{u}_{p_{m+1,n}}}{\Delta \bar{x}} + \frac{\bar{F} \, \bar{\rho}}{2}$$

$$C_n^{**} = -A_n^{**}$$

$$D_n^{**} = \frac{1}{2 \Delta \bar{x}} (\bar{u}^2_{p_{m+1,n}} + \bar{u}^2_{p_{m,n}})$$

$$- \frac{\bar{v}_{m+1/2,n}}{4 \Delta \bar{y}} (\bar{u}_{p_{m,n+1}} - \bar{u}_{p_{m,n-1}})$$

$$+ \frac{\bar{F} \, \bar{\rho}}{2} (\bar{u}_{m+1,n} + \bar{u}_{m,n} - \bar{u}_{p_{m,n}})$$

Boundary conditions (2.16) to (2.20) become

$$\bar{u}_{m,1} = \bar{v}_{m,1} = 0 \quad \text{for} \quad m \geq 1 \quad (2.42)$$

$$\bar{u}_{m,N} = \bar{u}_{p_{m,N}} = \bar{\rho}_{p_{m,N}} = 1 \quad \text{for} \quad m \geq 1 \quad (2.43)$$

$$\bar{u}_{p_{m+1,1}} = \bar{u}_{p_{m,1}} - \bar{F} \, \bar{\rho} \, \Delta \bar{x} \quad \text{for} \quad m \geq 1 \quad (2.44)$$

$$\bar{\rho}_{p_{m+1,1}} \bar{u}_{p_{m+1,1}} = \bar{\rho}_{p_{m,1}} \, \bar{u}_{p_{m,1}} \quad \text{for} \quad m \geq 1 \quad (2.45)$$

where $n = 1$ represents the plate surface and $n = N$ the outer edge of the boundary layer.

With prescribed initial values [eqs. (2.21) to (2.24) and eqs. (2.27) to (2.30)] and boundary conditions (2.42) to (2.45), eqs. (2.38) to (2.41) are solved recursively at each step in x-direction. The set of eqs. (2.39) for $n = 2$ to $N-1$ is solved first by matrix inversion method. In the beginning, the values of $\bar{u}_{m+1,n}$; $\bar{u}_{p_{m+1,n}}$; $\bar{\rho}_{p_{m+1,n}}$ and $\bar{v}_{m+1/2,n}$ occuring in the coefficients $A_n$, $B_n$, $C_n$ and $D_n$ at the $(m+1)^{th}$ step are replaced by the corresponding values at the $m^{th}$ step. The set of eqs. (2.39) is thus rendered linear and is solved by matrix inversion method. In later iterations, $\bar{u}_{m+1,n}$ in the coefficients are replaced by the values obtained in the previous iteration, while the values of $\bar{v}_{m+1/2,n}$, $\bar{u}_{p_{m+1,n}}$ and $\bar{\rho}_{p_{m+1,n}}$ in the coefficients $A_n$, $B_n$, $C_n$ and $D_n$ are kept constant. Iterations are repeated till the difference between the values of $\bar{u}$ at the grid points in $(m+1)^{th}$ step in two successive iterations is less than $10^{-4}$. Generally three iterations are sufficient to get the converged values of $\bar{u}$. Next the set of eqs. (2.38) for $n = 2, \ldots N$ for $\bar{v}$ is solved explicitly.

The values of $\bar{u}$, $\bar{v}$ thus obtained are substituted in the set of eqs. (2.41) for $n = 2, \ldots N-1$ and is solved implicitly by matrix inversion method. Similar iterative procedure

(used for $\bar{u}$) is used to get the converged value of $\bar{u}_p$. The set of eqs. (2.40) for $n = 2, \ldots N-1$ is linear and is solved implicitly for $\bar{\rho}_p$. This cycle of operation is repeated until the maximum difference in the successive values of a variable at each grid point is less than $10^{-4}$. Similar operations are carried out at each step in x-direction.

We then calculate the skin friction coefficient by the following formula

$$C_f = \left.\frac{\mu\frac{\partial u}{\partial y}}{\frac{1}{2}\rho u_\infty^2}\right|_{y=0} = \frac{2\frac{\partial \bar{u}}{\partial \bar{y}}}{\sqrt{R_e}} = \frac{2\bar{u}_{m+1,2}}{\Delta\bar{y}\sqrt{R_e}} \qquad (2.46)$$

As the integration is carried downstream, the step size $\Delta\bar{x}$ is increased by .01 after every 25 steps till $\Delta\bar{x} = .1$ is reached. Thereafter, it remains constant. $\Delta\bar{y} = .2$ is kept constant. At some point in the downstream direction, the boundary layer edge lies outside the range of integration. At this point, the difference in the values of $\bar{u}$, $\bar{u}_p$ and $\bar{\rho}_p$ at $N^{th}$ and $(N-1)^{th}$ point becomes greater than $10^{-3}$ and the range of integration in the y-direction is increased by 10 steps (Fig. 1). The integration is carried on an IBM 7044 computer of I.I.T. Kanpur. Depending upon the severity of parameters, it takes 3 minutes to 20 minutes of computer time.

## 2.4 Discussion of the results

We present here some of the characteristic features of gas-particulate flow in the boundary layer. The results of

the present analysis are compared with the results of Tabakoff and Hamed (1972). For comparison, following values of the parameters are used

$$u_\infty = 60.96 \text{ m/sec.}$$
$$\rho = .9752 \text{ kg./m}^3$$
$$\mu = 1.5415 \times 10^{-5} \text{ kg./m}^3$$
$$\rho_{sp} = 801.0,\ 1602.0,\ 2403.0,\ 8010.0 \text{ kg./m}^3$$
$$\alpha = 0.1,\ 0.2,\ 0.3,\ 0.4$$
$$d = 50\,\mu,\ 100\,\mu,\ 250\,\mu$$
$$L = .3048 \text{ m}$$

In Fig. 2, we have compared the skin friction coefficient predicted by the present method with the results obtained by Tabakoff and Hamed (1972). The present results are computed with the initial profiles given in eqs. (2.21) to (2.24) and eqs. (2.27) to (2.30) for $\rho_{sp} = 2403$ kg./m$^3$, $\alpha = 0.2$ and $d = 100\,\mu$. We note that both the initial profiles predict the same values of $C_f$. We further notice that the velocity profiles for the two phases are also identical for the two different initial profiles used in the computation. We thus surmise that the change in the initial profiles has no effect on the flow structure downstream. In the following results, we have used the initial profiles provided by Jain and Ghosh (1979). This figure also indicates that the present results predict a lower value of $C_f$ than the results of Tabakoff and Hamed (1972). For the prescribed values of the

parameters, the present results are obtained upto $\bar{x} = 17.1$. At $\bar{x} = 17.1$, the surface particulate density becomes excessively large and the analysis fails at this point. For the same values of the parameters, the approximate results of Tabakoff and Hamed (1972) hold upto $\bar{x} = 13.7$. Beyond this point, oscillations in the boundary layer thickness and skin-friction are set in and the results fail thereafter. This shows that present results have greater range of validity.

In Figs. 3 and 4, we have compared the profiles for $\bar{u}$ and $\bar{u}_p$ from the present calculations with the corresponding results of Tabakoff and Hamed (1972) for $\rho_{sp} = 2403$ kg./m$^3$, $\alpha = 0.2$ and $d = 100\ \mu$. Fig. 2 shows that at $\bar{x} = 6.0$, gas phase velocity profiles predicted by the present method and the approximate method agree reasonably well. Present method predicts velocity profile that lies slightly above the approximate method of Tabakoff and Hamed (1972). In figure 3, present results show that $\bar{u}_p$ increases monotonically from the value at the wall to its asymptotic value unity at the outer edge of the boundary layer while the approximate method of Tabakoff and Hamed (1972) gives the velocity profile for $\bar{u}_p$ which first decreases from its surface value and then increases to approach unity at the boundary layer edge. At $\bar{x} = 13.0$, particle velocity profile becomes negative in the boundary layer. This variation of $\bar{u}_p$ is not physically plausible. In this respect, present method seems to predict a correct $\bar{u}_p$-profile.

In Figs. 5 and 6, we find the effect of changes in $\bar{F}$ on $C_f$ and $\bar{u}_p$ profiles. Fig. 5 shows that $C_f$ increases as $\bar{F}$ increases. Fig. 6 shows that $\bar{u}_p$ increases as $\bar{F}$ decreases.

## Part II

2.5 The problem of gas-particulate boundary layer flow is reconsidered to study the effect of the compatibility conditions for particle phase on structure and surface characteristics of the flow. References are available in the literature which use and do not use the compatibility conditions for the particle phase in the calculation of boundary layer flow. Marble (1962) developed the series solution for incompressible gas phase and compressible particle phase without the use of compatibility conditions. His analysis indicated that the particle velocity on the surface remained zero upto the first order of the series solution. Singleton (1965) extended the analysis with compressible gas phase and obtained series solutions valid near the leading edge and far from the leading edge of the flat plate. Soo (1968) obtained the series solution using compatibility conditions on the flat plate by assuming $\rho_p/\rho \ll 1$. This assumption neglected the interaction term from the momentum equation of the gas phase and he could consider the Blasius solution for the gas phase. Far from the leading edge where particulate velocity is zero, he incorporated the diffusion of the particles in order to

avoid their accumulation on the surface. Far from the leading edge, this analysis predicted that $\rho_p$ decreased in the boundary layer and attained the value less than the free stream value. Soo (1967) solved the problem for incompressible gas phase by momentum integral method using linear profiles for the gas and for the particle velocities. Tabakoff and Hamed (1972) modified the procedure by taking the fourth degree profile for the gas phase and similar profile for the particle phase. Both of these investigators used the compatibility conditions for the particle phase and obtained a linear variation of the surface particulate velocity with the longitudinal distance 'x' on the plate. Jain and Ghosh (1979) developed an integral method dropping the compatibility conditions and obtained the solution for all values of the longitudinal distance 'x' on the plate. The present authors attempted to analyse the effect of the use of compatibility conditions in solving boundary layer equations on the flow structure and surface characteristics.

We solve gas-particulate boundary layer equations with and without the use of compatibility conditions by Crank-Nicholson scheme of finite differencing. We find that in both the cases, surface particulate velocity varies linearly with the longitudinal distance 'x'. With compatibility conditions, particulate density inside the boundary layer decreases from its surface value to a value much below the

free stream condition and then increases to attain the free stream value while without the use of compatibility conditions, the density decreases monotonically from its surface value to the free stream value. In this way, the results without compatibility conditions are superior to the results with compatibility conditions. For other variables of the flow, the results are similar but differ slightly in magnitude. In the absence of experimental data, it is difficult to say which of the two results is accurate.

## 2.6 Governing equations and boundary conditions

Equations governing the 2-dimensional steady gas particulate boundary layer flow on a flat plate are the following

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{2.47}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F}\,\bar{\rho}_p(\bar{u}-\bar{u}_p) \tag{2.48}$$

$$\frac{\partial}{\partial \bar{x}}(\bar{\rho}_p\,\bar{u}_p) + \frac{\partial}{\partial \bar{y}}(\bar{\rho}_p\,\bar{v}_p) = 0 \tag{2.49}$$

$$\bar{u}_p\frac{\partial \bar{u}_p}{\partial \bar{x}} + \bar{v}_p\frac{\partial \bar{u}_p}{\partial \bar{y}} = \bar{F}\,\bar{\rho}(\bar{u}-\bar{u}_p) \tag{2.50}$$

$$\bar{u}_p\frac{\partial \bar{v}_p}{\partial \bar{x}} + \bar{v}_p\frac{\partial \bar{v}_p}{\partial \bar{y}} = \bar{F}\,\bar{\rho}(\bar{v}-\bar{v}_p) \tag{2.51}$$

Here $\bar{v}_p$ is the normal component of velocity of the particle phase and the rest of the variables are the same as in (2.11).

For the analysis with compatibility conditions $v_p = v$ is assumed and then eqs. (2.47) to (2.50) after replacing $\bar{v}_p$ by $\bar{v}$ are solved for $\bar{u}$, $\bar{v}$, $\bar{u}_p$ and $\bar{\rho}_p$. Without compatibility conditions, we assume that $v_p \neq v$ and eqs. (2.47) to (2.51) are solved for $\bar{u}, \bar{v}, \bar{u}_p, \bar{v}_p$ and $\bar{\rho}_p$. For some unknown reasons, without using the compatibility conditions, we could not get the solution of eqs. (2.47) to (2.51) when $v_p = v$. As stated in chapter I, § 1.4, it is presumed that $\bar{v} = \bar{v}_p = O(\delta)$ and as such its retaining and dropping will not change the results substantially.

Boundary conditions

I Without compatibility conditions

(i) At $\bar{y} = 0$, $\bar{u} = \bar{v} = \bar{v}_p = 0$ (2.52)

(ii) As $\bar{y} \to \infty$, $\bar{u} = \bar{u}_p = \bar{\rho}_p = 1$ (2.53)

II With compatibility conditions

(i) At $\bar{y} = 0$, $\bar{u} = \bar{v} = 0$ (2.54)

(ii) $\frac{\partial}{\partial \bar{x}} (\bar{\rho}_p \bar{u}_p) \Big|_{\bar{y}=0} = 0$ (2.55)

(iii) $\left[\frac{\partial \bar{u}_p}{\partial \bar{x}}\right]_{\bar{y}=0} = -\bar{F}\,\bar{\rho}$ (2.56)

(iv) As $\bar{y} \to \infty$, $\bar{u} = \bar{u}_p = \bar{\rho}_p = 1$ (2.57)

The initial profiles are taken from eqs. (2.27) to (2.30) at $\bar{x} = 0.1$.

## 2.7 Solution of equations (2.47) to (2.51) without compatibility conditions

We divide the range of integration into grid points as defined in § 2.3 of part I and solve eqs. (2.47) to (2.51) with boundary conditions (2.52) to (2.53) by finite difference method due to Crank-Nicholson. The various terms are approximated as follows:

$$\bar{u}_{m+1/2,n} = \frac{1}{2}(\bar{u}_{m+1,n} - \bar{u}_{m,n}) \tag{2.58}$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_{m+1/2,n} = \frac{1}{\Delta \bar{x}}(\bar{u}_{m+1,n} - \bar{u}_{m,n}) \tag{2.59}$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{m+1/2,n} = \frac{1}{4\Delta \bar{y}}(\bar{u}_{m+1,n+1} - \bar{u}_{m+1,n-1} + \bar{u}_{m,n+1} - \bar{u}_{m,n-1}) \tag{2.60}$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_{m+1/2,n+1/2} = \frac{1}{\Delta \bar{x}}(\bar{u}_{m+1,n+1/2} - \bar{u}_{m,n+1/2}) \tag{2.61}$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{m+1/2,n+1/2} = \frac{1}{\Delta \bar{y}}(\bar{u}_{m+1/2,n+1} - \bar{u}_{m+1/2,n}) \tag{2.62}$$

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_{m+1/2,n} = \frac{1}{2(\Delta \bar{y})^2}(\bar{u}_{m+1,n+1} - 2\bar{u}_{m+1,n} + \bar{u}_{m+1,n-1} + u_{m,n+1} - 2\bar{u}_{m,n} + \bar{u}_{m,n-1}) \tag{2.63}$$

Using (2.58) to (2.63) and similar expressions for other variables $\bar{v}, \bar{u}_p, \bar{v}_p$ and $\bar{\rho}_p$, eqs. (2.47) to (2.51) are differenced at the points $(m+\frac{1}{2}, n-\frac{1}{2})$; $(m+\frac{1}{2}, n)$; $(m+\frac{1}{2}, n+\frac{1}{2})$; $(m+\frac{1}{2}, n+\frac{1}{2})$ and $(m+\frac{1}{2}, n-\frac{1}{2})$ respectively. The resulting difference equations are:

$$\bar{v}_{m+1/2,n} = \bar{v}_{m+1/2,n-1} - \frac{\Delta\bar{y}}{2\Delta\bar{x}}\left(\bar{u}_{m+1,n} + \bar{u}_{m+1,n-1} - \bar{u}_{m,n} - \bar{u}_{m,n-1}\right) \tag{2.64}$$

$$A_n\,\bar{u}_{m+1,n+1} + B_n\,\bar{u}_{m+1,n} + C_n\,\bar{u}_{m+1,n-1} = D_n \tag{2.65}$$

where

$$A_n = \frac{\bar{u}_{m+1/2,n}}{4\Delta\bar{y}} - \frac{1}{2(\Delta\bar{y})^2}$$

$$B_n = \frac{\bar{u}_{m+1,n}}{\Delta\bar{x}} + \frac{1}{(\Delta\bar{y})^2} + \frac{\bar{F}}{2}\left(\bar{\rho}_{p_{m+1,n}} + \bar{\rho}_{p_{m,n}}\right)$$

$$C_n = \frac{\bar{v}_{m+1/2,n}}{4\Delta\bar{y}} - \frac{1}{2(\Delta\bar{y})^2}$$

$$D_n = \frac{1}{2\Delta\bar{x}}\left(\bar{u}^2_{m+1,n} + \bar{u}^2_{m,n}\right) - \frac{\bar{v}_{m+1/2,n}}{4\Delta\bar{y}}\left(\bar{u}_{m,n+1} - \bar{u}_{m,n-1}\right)$$

$$+ \frac{1}{2(\Delta\bar{y})^2}\left(\bar{u}_{m,n+1} - 2\bar{u}_{m,n} + \bar{u}_{m,n-1}\right)$$

$$- \frac{\bar{F}}{4}\left(\bar{\rho}_{p_{m+1,n}} + \bar{\rho}_{p_{m,n}}\right)\left(\bar{u}_{m,n} - \bar{u}_{p_{m+1,n}} - \bar{u}_{p_{m,n}}\right)$$

$$P_n\,\bar{\rho}_{p_{m+1,n+1}} + Q_n\,\bar{\rho}_{p_{m+1,n}} = R_n \tag{2.66}$$

where

$$P_n = \frac{\bar{u}_{p_{m+1,n+1}} + \bar{u}_{p_{m+1,n}}}{4\Delta\bar{x}} + \frac{\bar{v}_{p_{m+1,n+1}} + \bar{v}_{p_{m,n+1}}}{4\Delta\bar{y}}$$

$$Q_n = \frac{\bar{u}_{p_{m+1,n+1}} + \bar{u}_{p_{m+1,n}}}{4\Delta\bar{x}} - \frac{\bar{v}_{p_{m+1,n}} + \bar{v}_{p_{m,n}}}{4\Delta\bar{y}}$$

$$R_n = \bar{\rho}_{p_{m,n+1}}\left(\frac{\bar{u}_{p_{m,n+1}} + \bar{u}_{p_{m,n}}}{4\Delta\bar{x}} - \frac{\bar{v}_{p_{m+1,n+1}} + \bar{v}_{p_{m,n+1}}}{4\Delta\bar{y}}\right)$$

$$+ \bar{\rho}_{p_{m,n}}\left(\frac{\bar{u}_{p_{m,n+1}} + \bar{u}_{p_{m,n}}}{4\Delta\bar{x}} + \frac{\bar{v}_{p_{m+1,n}} + \bar{v}_{p_{m,n}}}{4\Delta\bar{y}}\right)$$

$$P_n^* \, \bar{u}_{p_{m+1,n+1}} + Q_n^* \, \bar{u}_{p_{m+1,n}} = R_n^* \tag{2.67}$$

where

$$P_n^* = \frac{\bar{u}_{p_{m+1,n+1}} + \bar{u}_{p_{m+1,n}}}{4\Delta\bar{x}} + \frac{\bar{F}\,\bar{\rho}}{4}$$

$$+ \frac{\bar{v}_{p_{m+1,n+1}} + \bar{v}_{p_{m+1,n}} + \bar{v}_{p_{m,n+1}} + \bar{v}_{p_{m,n}}}{8\Delta\bar{y}}$$

$$Q_n^* = \frac{\bar{u}_{p_{m+1,n+1}} + \bar{u}_{p_{m+1,n}}}{4\Delta\bar{x}} + \frac{\bar{F}\,\bar{\rho}}{4}$$

$$- \frac{\bar{v}_{p_{m+1,n+1}} + \bar{v}_{p_{m+1,n}} + \bar{v}_{p_{m,n+1}} + \bar{v}_{p_{m,n}}}{8\Delta\bar{y}}$$

$$R_n^* = \frac{1}{8\Delta\bar{x}} \left[ (\bar{u}_{p_{m+1,n+1}} + \bar{u}_{p_{m+1,n}})^2 + (\bar{u}_{p_{m,n+1}} + \bar{u}_{p_{m,n}})^2 \right]$$

$$- \frac{1}{8\Delta\bar{y}} (\bar{u}_{p_{m,n+1}} - \bar{u}_{p_{m,n}})(\bar{v}_{p_{m+1,n+1}} + \bar{v}_{p_{m,n+1}}$$

$$+ \bar{v}_{p_{m+1,n}} + \bar{v}_{p_{m,n}})$$

$$+ \frac{\bar{F}\,\bar{\rho}}{4} (\bar{u}_{m+1,n+1} + \bar{u}_{m+1,n} + \bar{u}_{m,n+1} + \bar{u}_{m,n}$$

$$- \bar{u}_{p_{m,n+1}} - \bar{u}_{p_{m,n}})$$

$$P_n^{**} \, \bar{v}_{p_{m+1,n}} + Q_n^{**} \, \bar{v}_{p_{m+1,n-1}} = R_n^{**} \tag{2.68}$$

where

$$P_n^{**} = \frac{1}{8\Delta\bar{x}} (\bar{u}_{p_{m+1,n}} + \bar{u}_{p_{m+1,n-1}} + \bar{u}_{p_{m,n}} + \bar{u}_{p_{m,n-1}})$$

$$+ \frac{\bar{v}_{p_{m+1,n}}}{8\Delta\bar{y}} + \frac{\bar{v}_{p_{m,n}}}{4\Delta\bar{y}} + \frac{\bar{F}\,\bar{\rho}}{4}$$

$$Q_n^{**} = \frac{1}{8\Delta\bar{x}}\left(\bar{u}_{p_{m+1,n}} + \bar{u}_{p_{m+1,n-1}} + \bar{u}_{p_{m,n}} + \bar{u}_{p_{m,n-1}}\right) - \frac{\bar{v}_{p_{m+1,n-1}}}{8\Delta\bar{y}} - \frac{\bar{v}_{p_{m,n-1}}}{4\Delta\bar{y}} + \frac{\bar{F}\,\bar{\rho}}{4}$$

$$R_n^{**} = \frac{\bar{F}\,\bar{\rho}}{4}\left[(\bar{v}_{m+1/2,n} + \bar{v}_{m+1/2,n-1}) - \bar{v}_{p_{m,n-1}} - \bar{v}_{p_{m,n}}\right] + \frac{1}{8\Delta\bar{y}}\left[(\bar{v}_{p_{m,n}} + \bar{v}_{p_{m,n-1}})(\bar{u}_{p_{m+1,n}} + \bar{u}_{p_{m+1,n-1}} + \bar{u}_{p_{m,n-1}} + \bar{u}_{p_{m,n}})\right] + \frac{1}{8\Delta\bar{y}}\left(\bar{v}^2_{p_{m,n-1}} - \bar{v}^2_{p_{m,n}}\right)$$

Boundary conditions in difference form become the following:

I Without compatibility conditions

$$\bar{u}_{m,1} = \bar{v}_{m,1} = \bar{v}_{p_{m,1}} = 0 \quad \text{for} \quad m \geq 1 \tag{2.69}$$

$$\bar{u}_{m,N} = \bar{u}_{p_{m,N}} = \bar{\rho}_{p_{m,N}} = 1 \quad \text{for} \quad m \geq 1 \tag{2.70}$$

II With compatibility conditions

$$\bar{u}_{m,1} = \bar{v}_{m,1} = 0 \quad \text{for} \quad m \geq 1 \tag{2.71}$$

$$\bar{\rho}_{p_{m+1,1}}\bar{u}_{p_{m+1,1}} = \bar{\rho}_{p_{m,1}}\,\bar{u}_{p_{m,1}} \quad \text{for} \quad m \geq 1 \tag{2.72}$$

$$\bar{u}_{p_{m+1,1}} = \bar{u}_{p_{m,1}} - \bar{F}\,\bar{\rho}\,\Delta\bar{x} \quad \text{for} \quad m \geq 1 \tag{2.73}$$

$$\bar{u}_{m,N} = \bar{u}_{p_{m,N}} = \bar{\rho}_{p_{m,N}} = 1 \quad \text{for} \quad m \geq 1 \tag{2.74}$$

With prescribed initial values at $\bar{x}_1 = .1$, system of eqs. (2.64) to (2.68) with boundary conditions (2.69) to (2.70) is solved recursively at the point $(\bar{x}_1 + \Delta\bar{x})$. System of eqs. (2.65) for $n = 2, \ldots N-1$ for $\bar{u}$ is solved first. In the beginning, the values of $\bar{u}_{m+1,n}$, $\bar{u}_{p_{m+1,n}}$; $\bar{\rho}_{p_{m+1,n}}$ and $\bar{v}_{m+1/2,n}$ occuring in the coefficients $A_n$, $B_n$, $C_n$, $D_n$ at the $(m+1)^{th}$ step are replaced by the corresponding values at the $m^{th}$ step. The set of eqs. (2.65) is thus rendered linear and is solved by matrix inversion method. In later iterations, values of $\bar{u}_{m+1,n}$ in the coefficients $A_n$, $B_n$, $C_n$ and $D_n$ are replaced by the values obtained in the previous iteration while the values of $\bar{v}_{m+1/2,n}$, $\bar{u}_{p_{m+1,n}}$ and $\bar{\rho}_{p_{m+1,n}}$ in the coefficients $A_n$, $B_n$, $C_n$ and $D_n$ are kept constant. Iterations are repeated till the difference between the values of $\bar{u}$ at the grid points in $(m+1)^{th}$ step in two successive iterations is less than $10^{-4}$. Generally three iterations are sufficient to get the converged values of $\bar{u}$. Next, we solve explicitly the set of eqs. (2.64) for $n = 2, \ldots N$ for $\bar{v}$. With values of $\bar{u}, \bar{v}$ thus obtained, the set of eqs. (2.68) for $n = 2, \ldots N$ for $\bar{v}_p$ is solved next. The values of $\bar{u}, \bar{v}_p$ thus obtained are substituted in the set of eqs. (2.67) for $n = 1, \ldots N-1$ and is solved explicitly to get the values of $\bar{u}_p$. In the end, we solve explicitly the set of eqs. (2.66) for $n = 1, \ldots N-1$ for $\bar{\rho}_p$. This cycle of operation is repeated till the values of the variables at different grid

points at $(m+1)^{th}$ step differ by less than $10^{-4}$ in successive cycles and the calculation proceeds to the next step. At each step, skin-friction coefficient $C_f$ is calculated by the formula given in eq. (2.46).

A constant step size of $\Delta\bar{x} = .01$ and $\Delta\bar{y} = .2$ is used. Both the step sizes are kept constant throughout the integration. When the computation proceeds in the downstream direction and the difference in $\bar{u}$, $\bar{u}_p$ and $\bar{\rho}_p$ at $N^{th}$ and $(N-1)^{th}$ grid points at any station $\bar{x}_m$ on the plate exceeds a prescribed value, the range of integration in the y-direction is increased so as to get a smooth variation of the dependent variables. Initially 40 steps in y-direction are used and when the boundary layer edge is reached, this range is increased by 10 steps. Depending on the severity of the variables, computer time on IBM 7044 computer varies from 10 to 45 minutes.

The solution of the equations with compatibility conditions is discussed in Part I of the chapter.

## 2.8 Discussion of the results

We compute the various flow variables for same values of the parameters considered in § 2.4 of Part I and the results obtained from the analysis without compatibility conditions are compared with the results with compatibility conditions discussed in Part I.

In Fig. 7, we compare the results of the skin-friction coefficient predicted by the analyses with and without compatibility conditions with the results for clean gas flow. We find that the presence of particles in the gas increases the value of skin-friction coefficient. Also, $C_f$ predicted with compatibility conditions has lower values than $C_f$ without compatibility conditions. In Fig. 8 we compare values of $\bar{u}_{p_w}$ obtained from the two analyses. It is surprising to note that the results obtained without the use of compatibility conditions are exactly the same as the values obtained with the use of compatibility conditions. Both the approaches give a linear variation of $\bar{u}_{pw}$ with the distance '$\bar{x}$' on the plate. At $\bar{x} = 17.1$, $\bar{u}_{pw} = 0$ and $\bar{\rho}_{pw}$ becomes excessively large. Beyond this point, oscillations in $\bar{\rho}_p$-profile are set up and the method fails to proceed further. In Fig. 9, we compare the values of $\bar{\rho}_{pw}$ calculated from both the analyses. The graph illustrates that the values of $\bar{\rho}_{pw}$ calculated from both the analyses agree upto $\bar{x} = 13.0$ and differ slightly thereafter. However, there is a sharp rise in the values of $\bar{\rho}_{pw}$ beyond $\bar{x} = 13.0$.

In Fig. 10, we compare $\bar{u}$-profile by both the analyses. We find that $\bar{u}$-profile obtained from the analysis without compatibility conditions lies slightly below the profile predicted by the analysis with compatibility conditions.

11 illustrates that the $\bar{u}_p$-profile obtained by using

compatibility conditions lies slightly above the profile obtained without using the compatibility conditions. This graph further shows the entirely different behaviour of particulate density in the boundary layer. Relaxing the compatibility conditions, the analysis predicts better structure of $\bar{\rho}_p$-profile. The values of $\bar{\rho}_p$ decreases monotonically from the surface value to the free stream value at the edge of the boundary layer. The analysis with compatibility conditions shows that $\bar{\rho}_p$ first decreases from the surface value and goes much below unity in the boundary layer and then it starts increasing and attains the value of unity at the boundary layer edge. Thus, the analysis without compatibility conditions predicts better structure of the flow than the corresponding results with compatibility conditions.

In Figs. 12 and 13, we give the variation of $\bar{u}_p$ and $\bar{\rho}_p$ with $\bar{F}$ when the compatibility conditions are **not** used. Fig. 12 shows that $\bar{u}_p$ decreases as $\bar{F}$ increases and Fig. 13 gives that $\bar{\rho}_p$ increases as $\bar{F}$ increases.

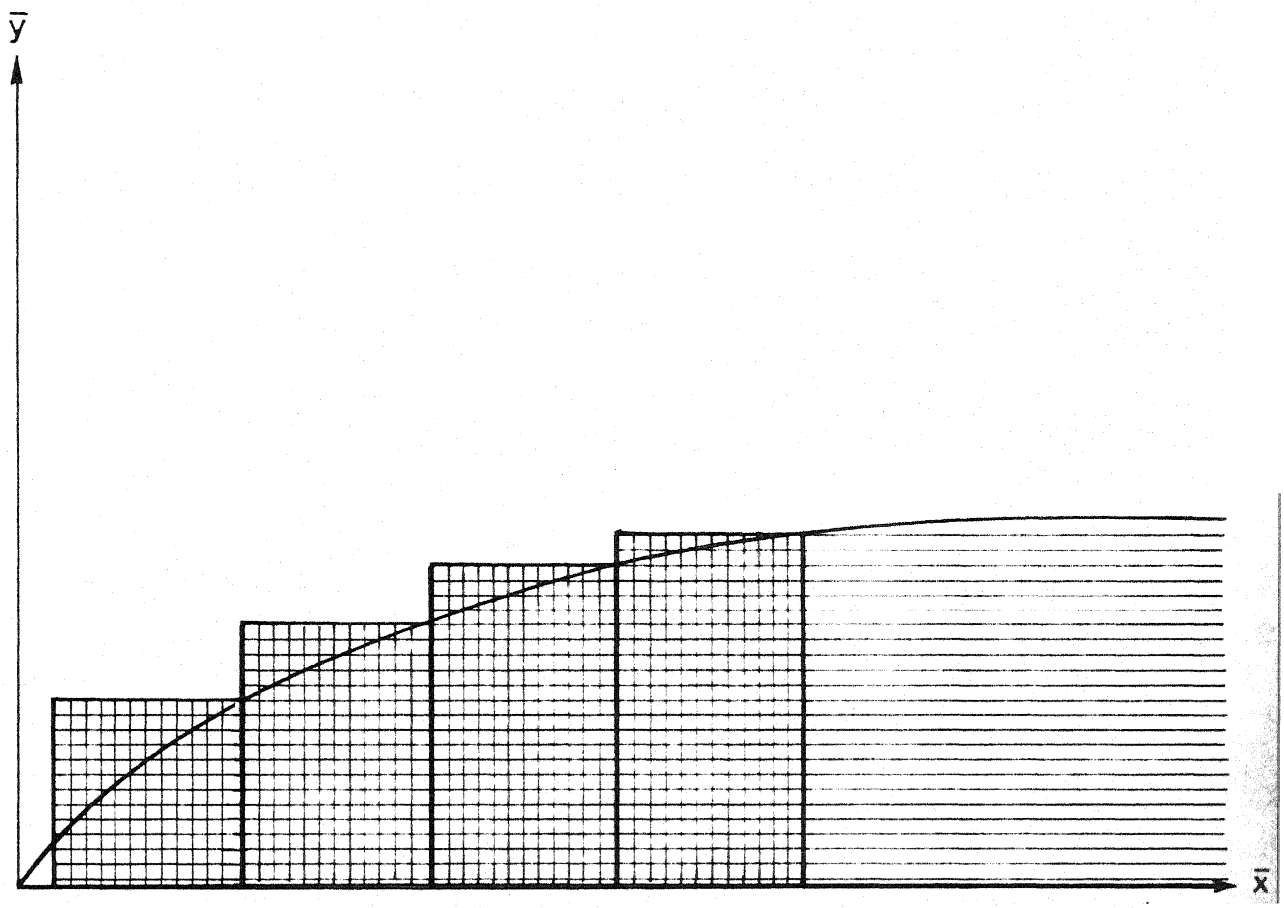



FIG. 1. GRID POINTS AND THE RANGE OF INTEGRATION.

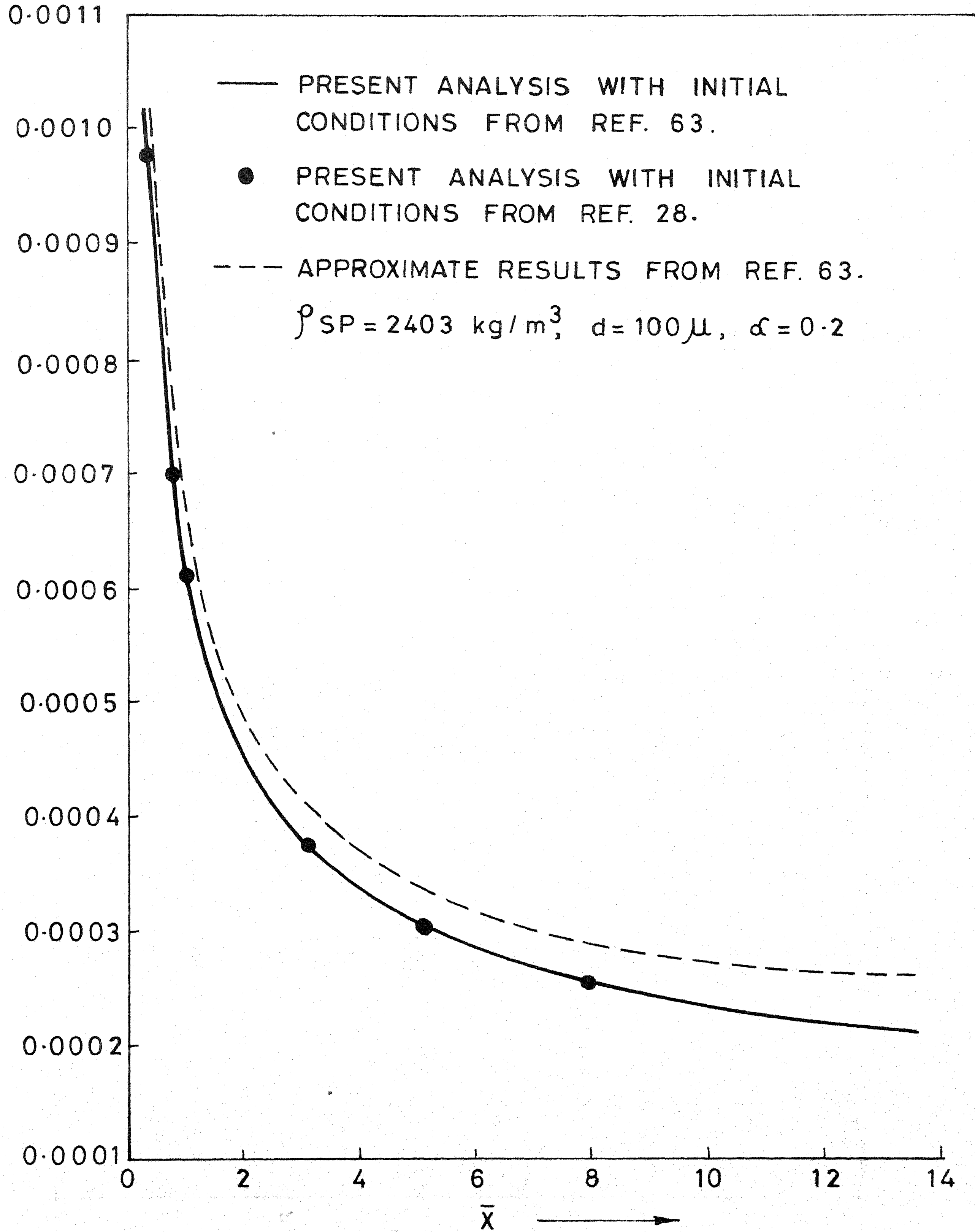



FIG. 2. COMPARISON OF SKIN FRICTION COEFFICIENT FOR GAS PARTICULATE FLOW.

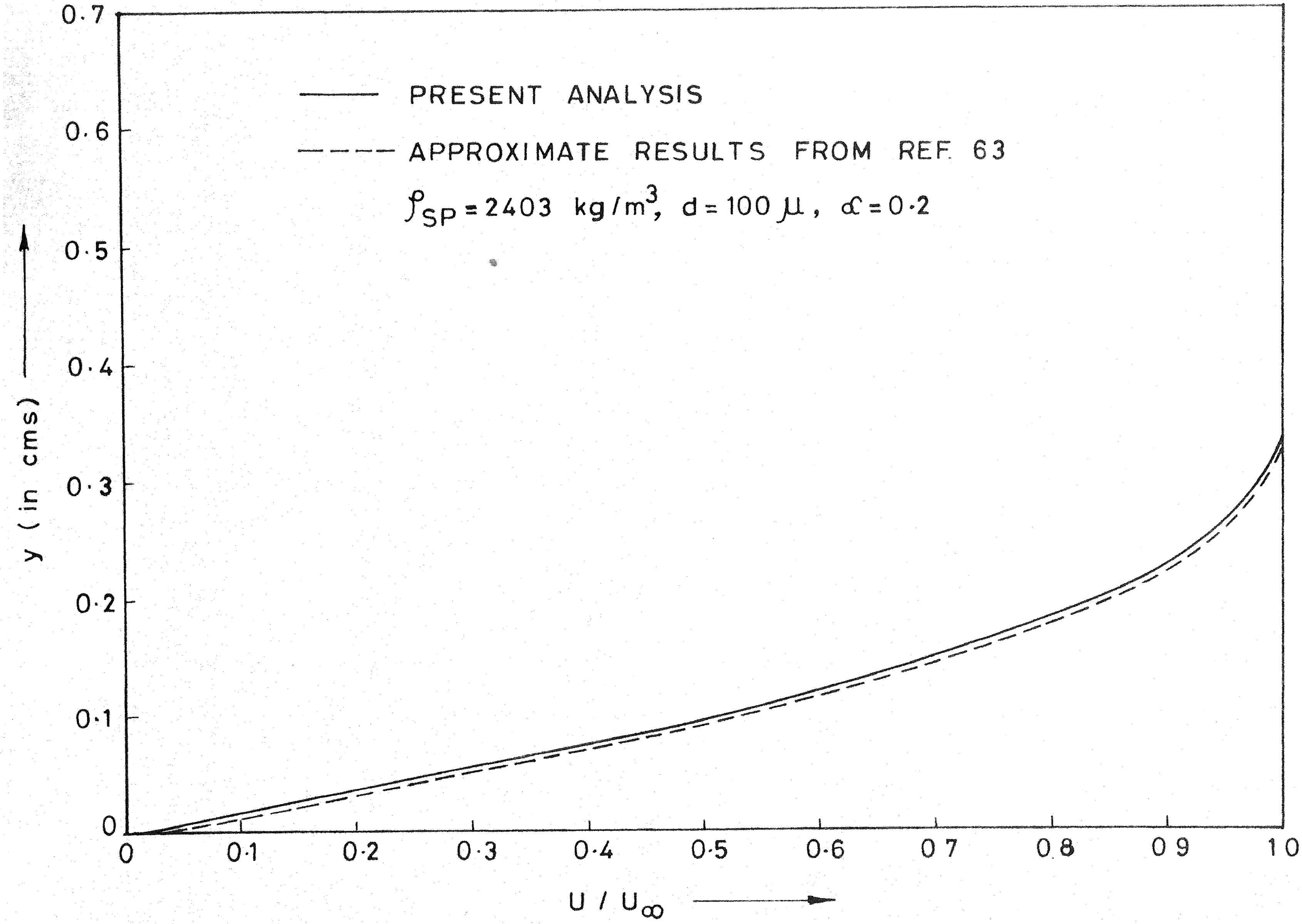



FIG. 3. GAS PHASE VELOCITY DISTRIBUTION ACROSS BOUNDARY

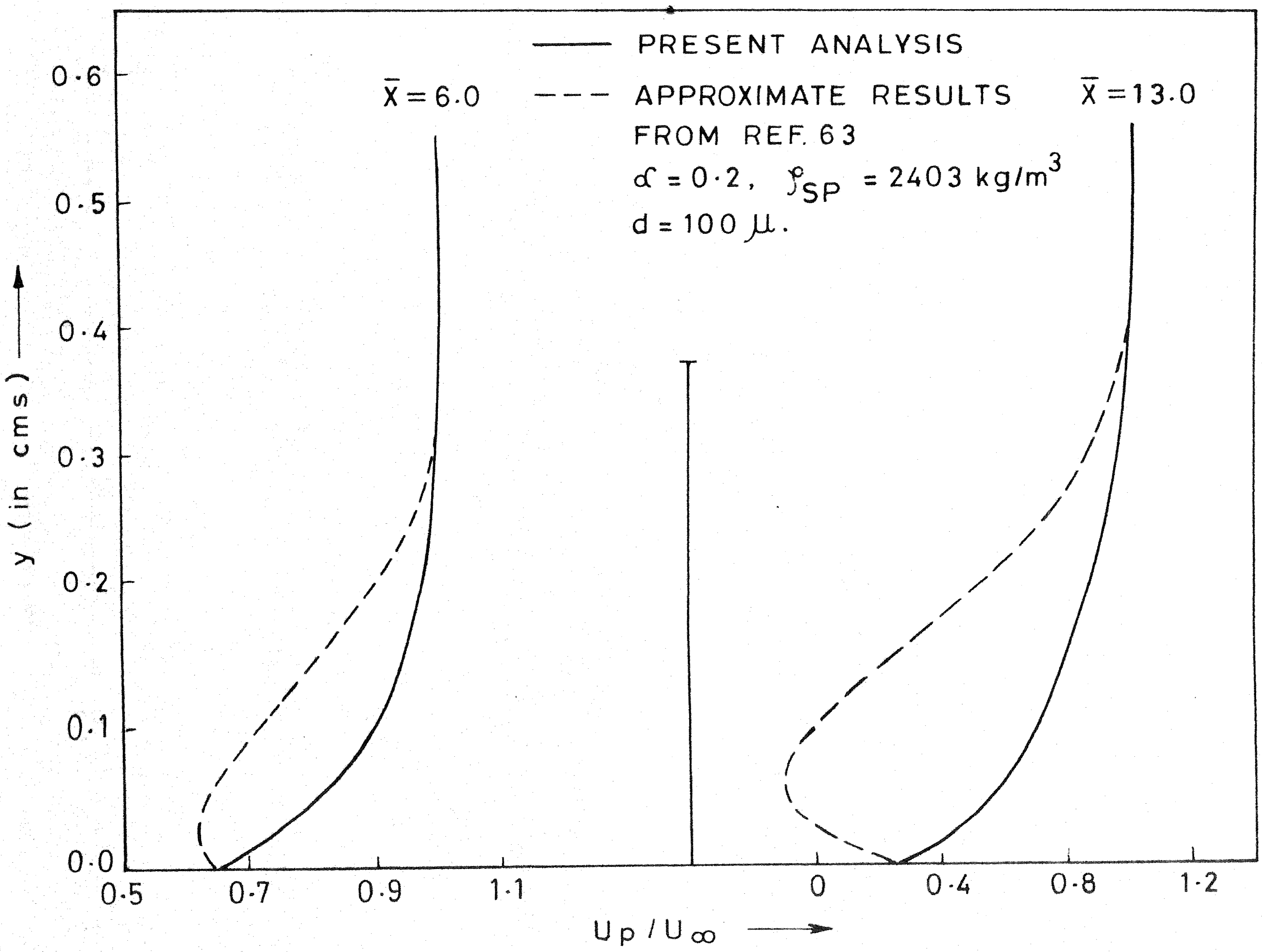
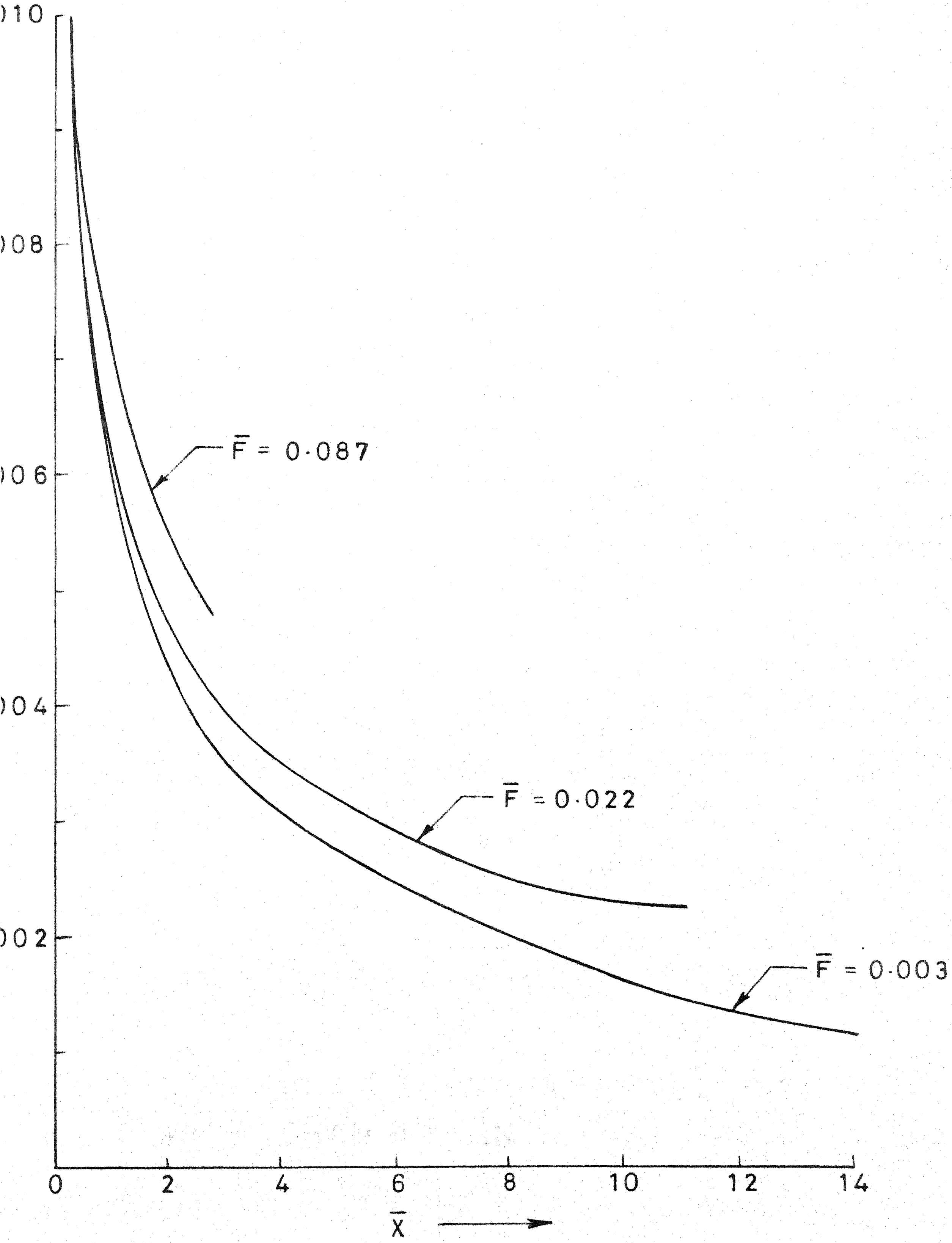



FIG. 4. PARTICULATE PHASE VELOCITY DISTRIBUTION ACROSS BOUNDARY LAYER.




VARIATION OF SKIN FRICTION COEFFICIENT WITH CHANGES IN $\bar{F}$.

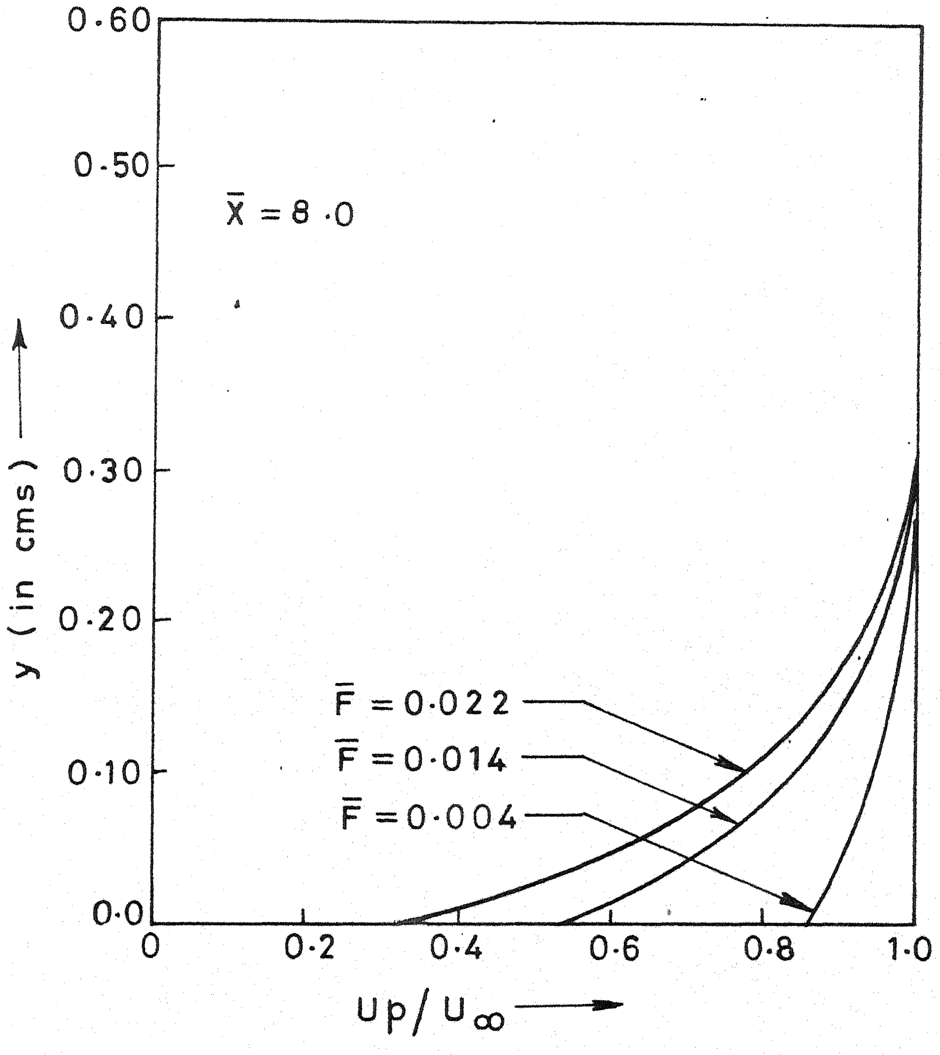



FIG. 6. VARIATION OF PARTICLE VELOCITY WITH CHANGES IN $\bar{F}$.

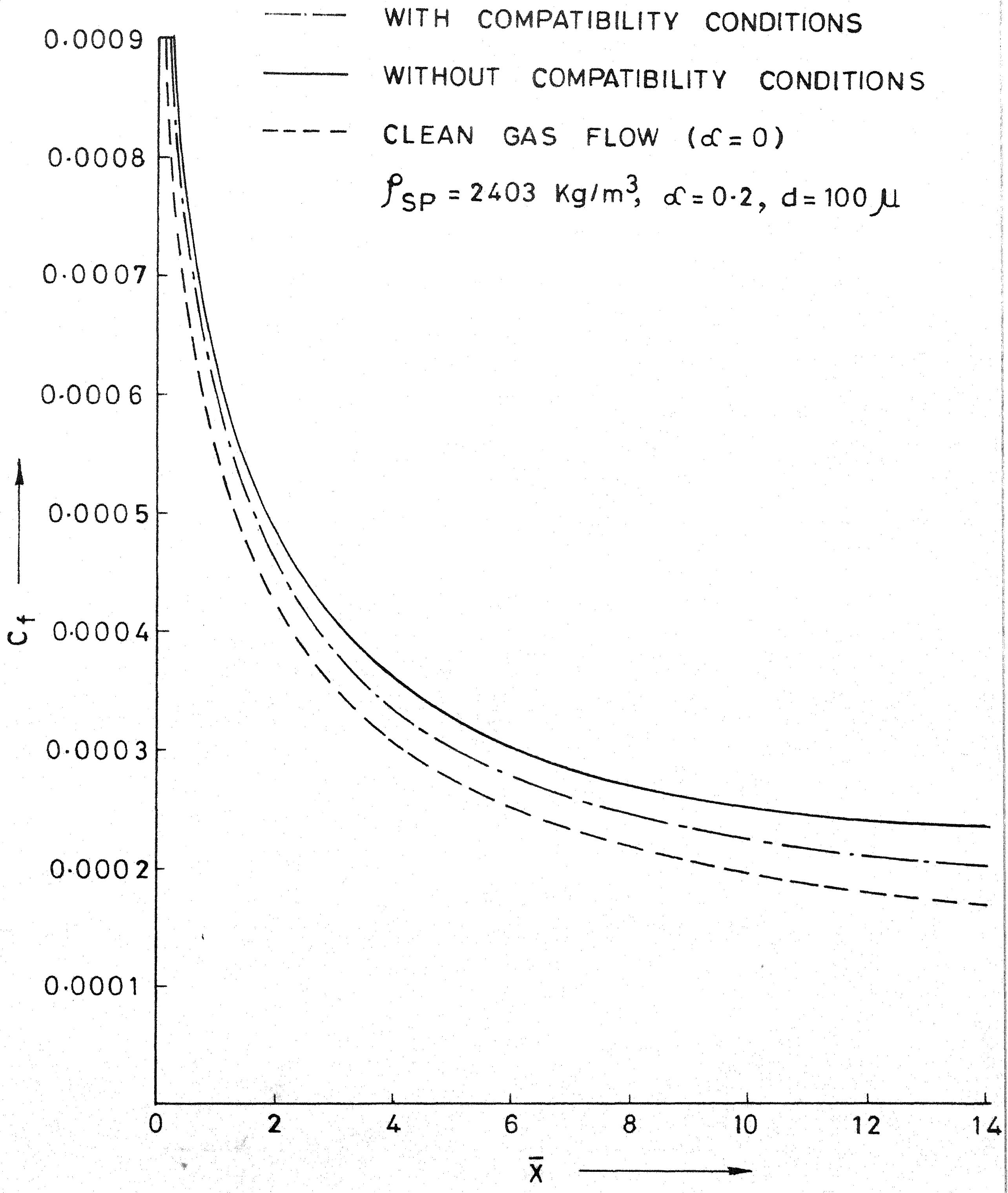



FIG. 7. COMPARISON OF SKIN FRICTION COEFFICIENT GAS PARTICULATE FLOW.

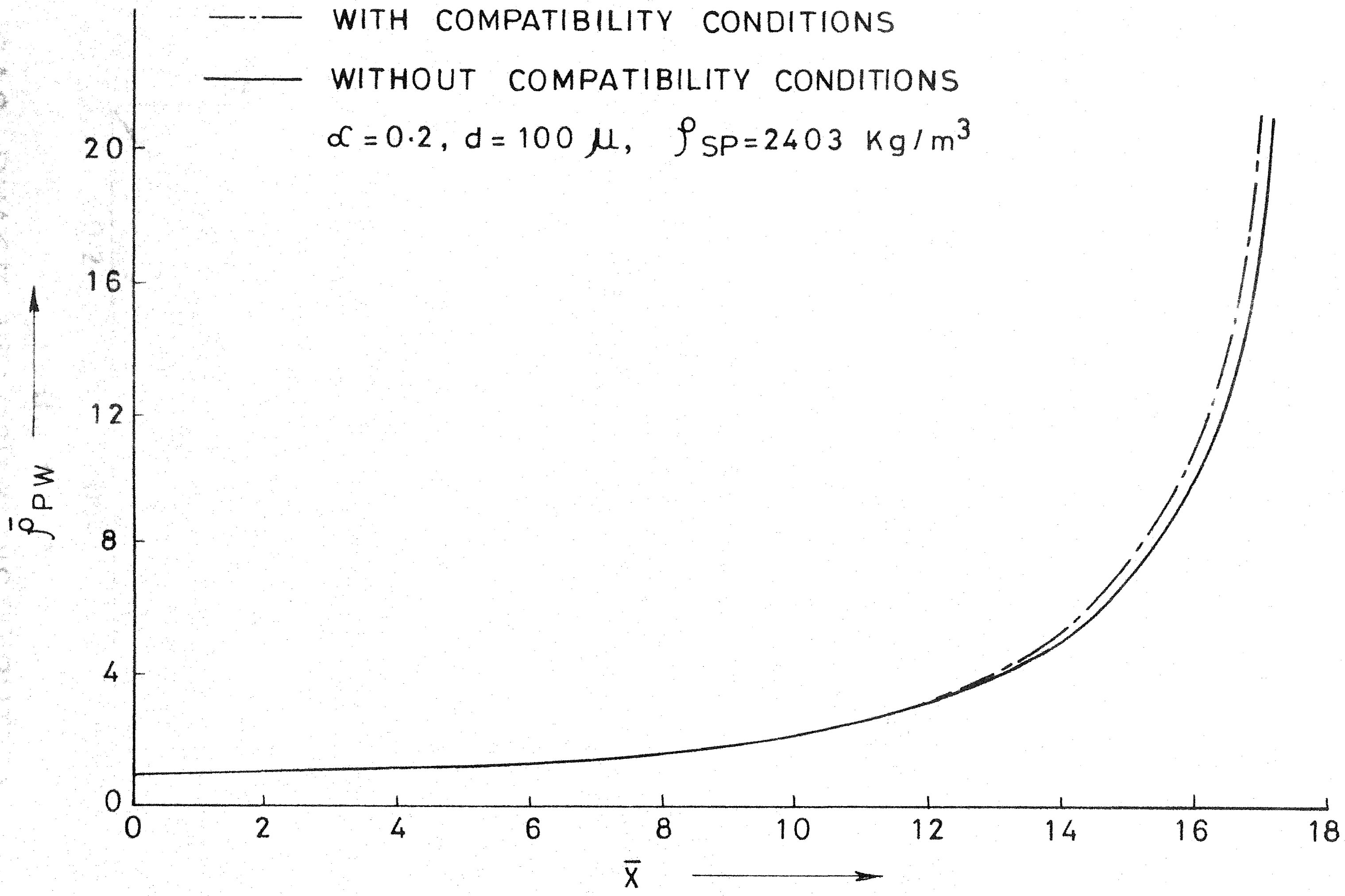
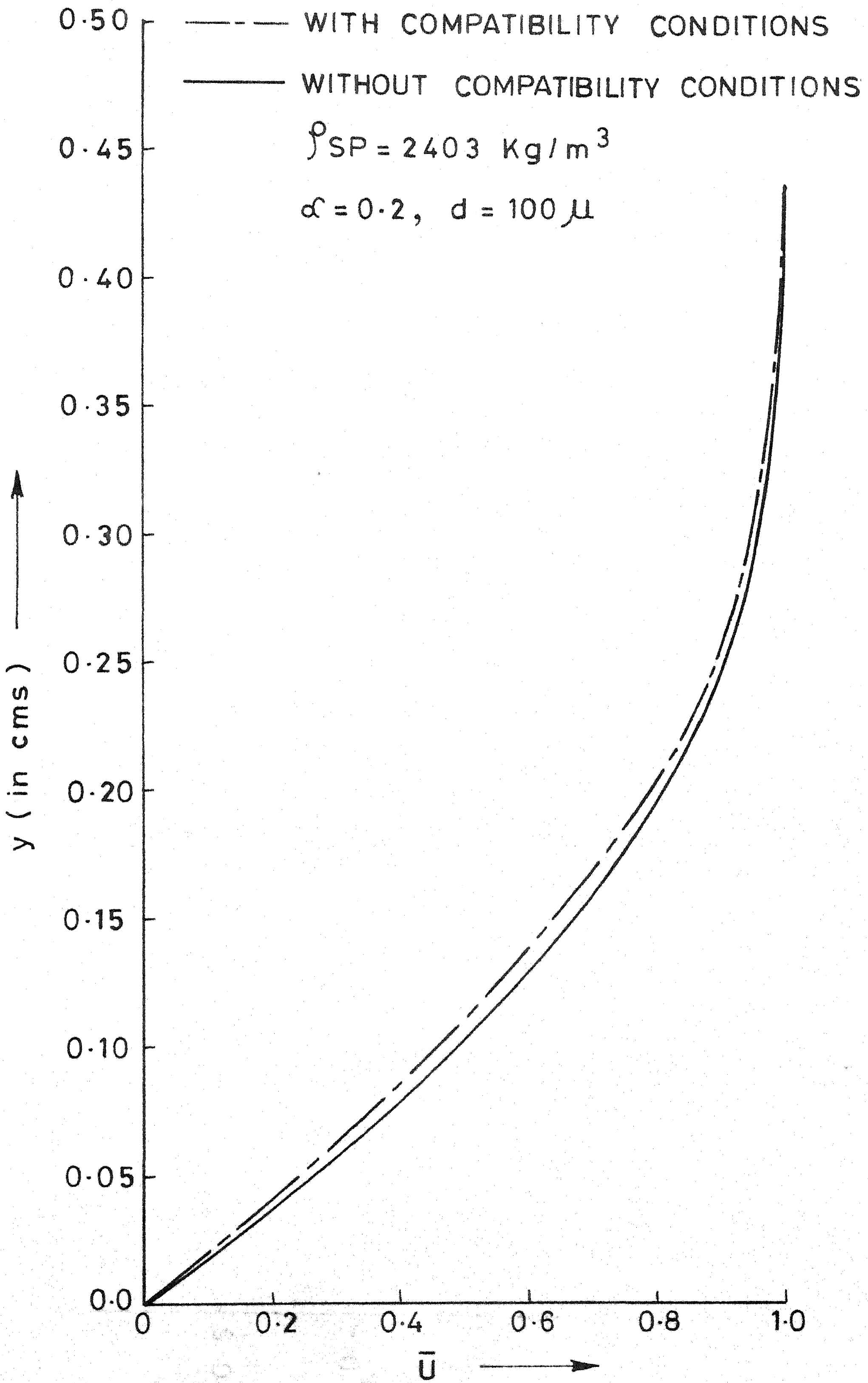



FIG. 9. SURFACE PARTICULATE DENSITY DISTRIBUTION.




IG. 10. GAS PHASE VELOCITY DISTRIBUTION ACROSS BOUNDARY LAYER AT $\bar{X} = 8{\cdot}0$

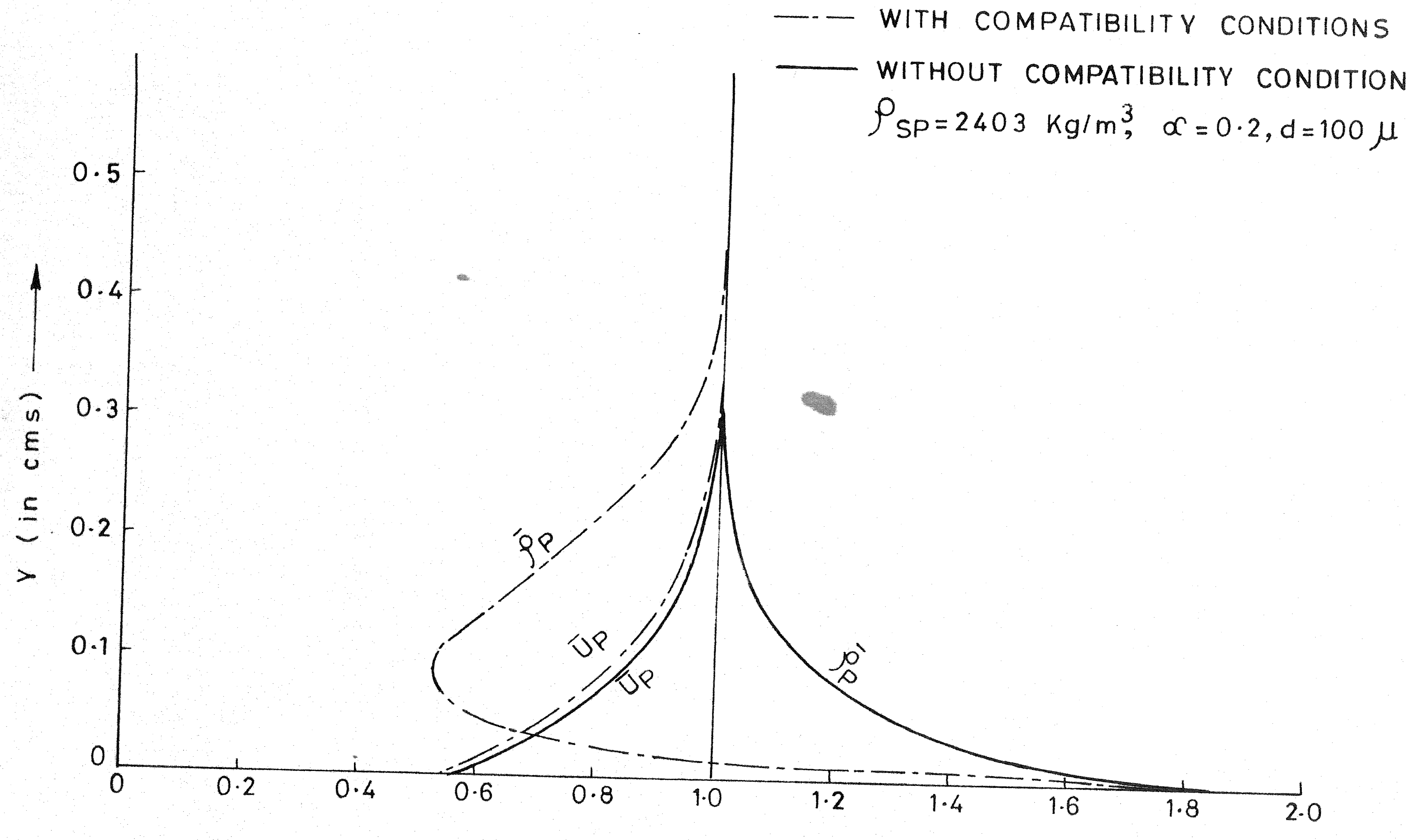



FIG. 11. VARIATION OF $\bar{U}_P$ AND $\bar{\rho}_P$ PROFILES IN THE BOUNDARY LAYER AT $\bar{X} = 8.0$

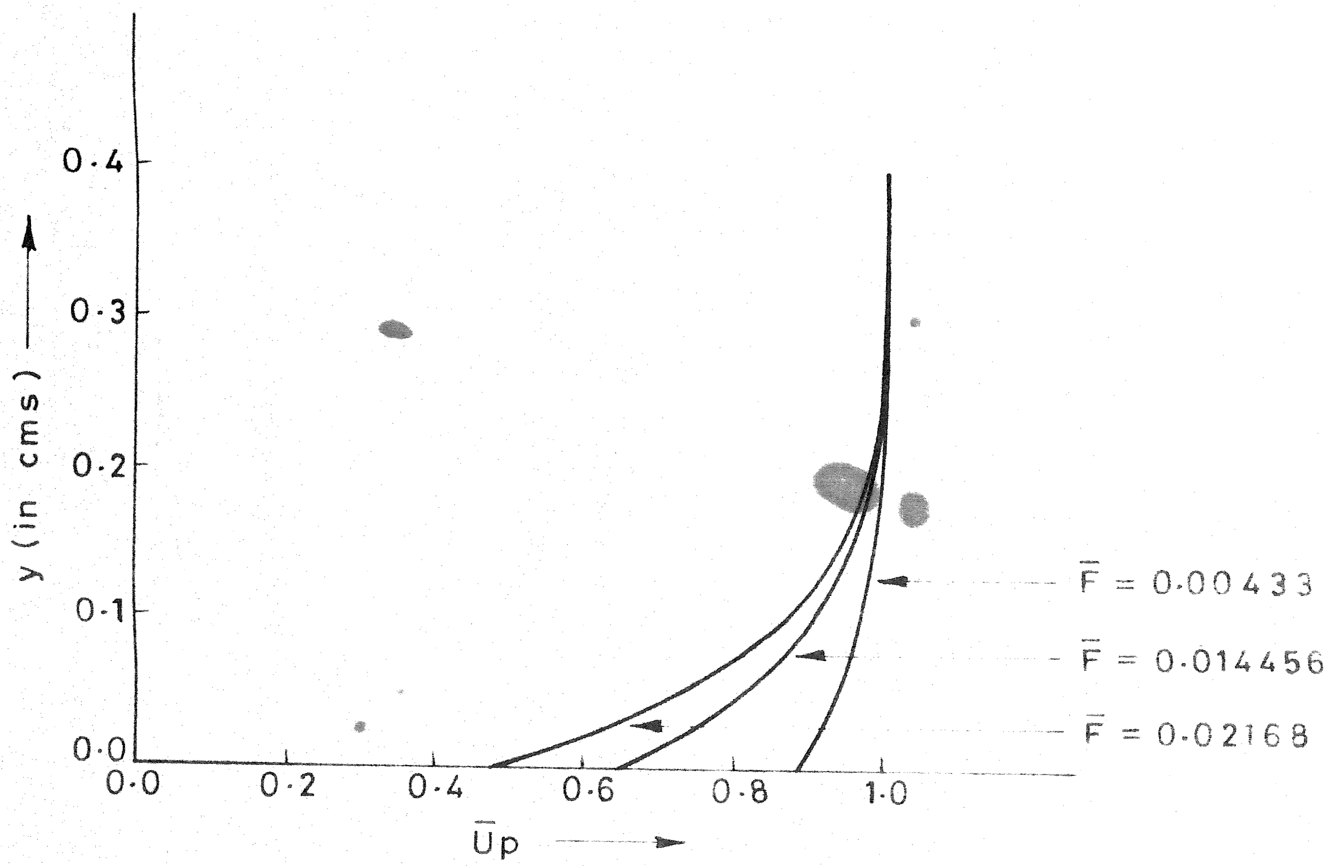



FIG. 12 VARIATION IN PARTICULATE VELOCITY PROFILE WITH CHANGES IN $\overline{F}$ AT $\overline{X} = 6{\cdot}0$.

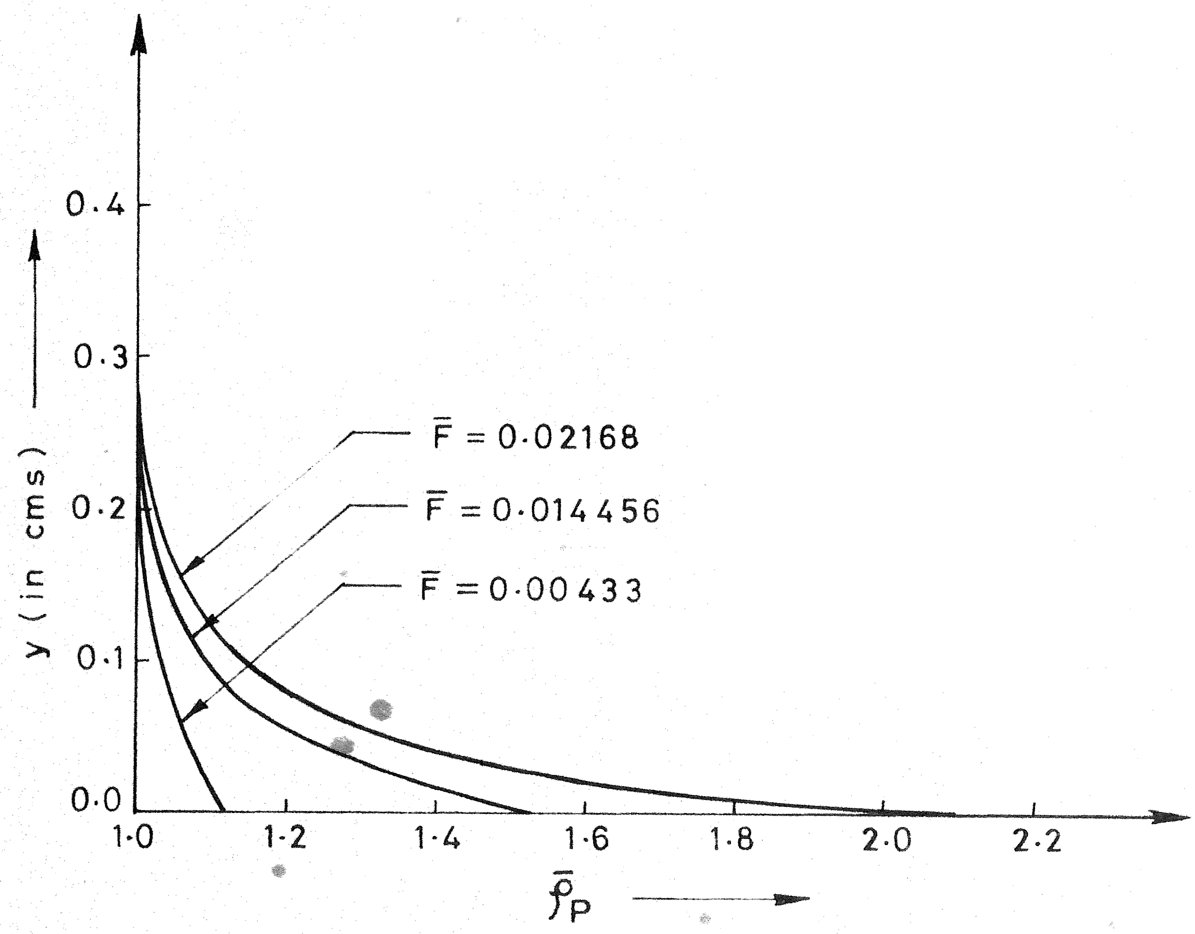



FIG. 13. VARIATION IN PARTICULATE DENSITY PROFILE WITH CHANGES IN $\bar{F}$ AT $\bar{X} = 6.0$.

# CHAPTER III

## GAS-PARTICULATE FLOW IN BETWEEN POROUS PARALLEL PLATES

### 3.1 Introduction

Closed form solutions for unsteady gas-particulate flow problems had been obtained by several authors. For example, Liu (1966) studied the flow induced in an incompressible dusty gas by an infinite flat plate oscillating in its own plane. He obtained analytical expressions for the velocity profiles for the gas and for the particle phases, shear stresses at the plate and discussed mechanical energy dissipation. He indicated that thickness of the viscous diffusion layer was decreased by the presence of the particles and the shear stresses at the plate were higher than the shear stresses for the gas phase only. In a later paper, Liu (1967) studied the flow induced by an infinite flat plate suddenly set into motion parallel to its own plane in an incompressible dusty gas. He gave approximate results for the velocity profiles in the form of series solutions valid for small times and for large times and calculated the corresponding skin-friction coefficients. His analysis indicated that for small times, viscous diffusion layer grows parabolically like $(\nu t)^{1/2}$ ( $\nu$ being the kinematic viscosity for the gas phase and $t$ the time variable), independent of the presence of the particle phase. For large times, viscous diffusion layer also

grows parabolically but as $(\bar{\nu}t)^{1/2}$ where $\bar{\nu}$ is the kinematic viscosity depending upon the viscosity of the gas but total density of the combined gas phase and solid particles.

Healy and Yang (1970) studied the oscillating two phase flows with viscous incompressible fluid and particle phases in a channel under the action of body and pressure forces. He showed that for the case of fixed body force acting on the fluid, the system tended to behave as a particle free fluid in equilibrium. It was also shown that the effects of the body and pressure forces were almost identical when particle material density is very much greater than the density of the fluid. When both the densities were approximately equal, the effects were significantly different. Ragland and Peddieson (1977) extended the work of Healy and Yang (1970) for finite volume fraction of the particles and obtained that the loading parameter $\kappa(= \frac{\rho_p}{\rho}$, $\rho_p$ and $\rho$ being the densities of the particle and gas phases respectively) had a much greater influence on the behaviour of the suspension than the effect due to the volume fraction $\phi$. Peddieson (1976) discussed five unsteady dusty gas flow problems in which the motion is induced in a semi-infinite mass of suspension by excitation at its boundaries. Such flows were shown to exhibit boundary layer behaviour.

In part I of the present chapter, we consider the steady gas particulate flow between porous infinite parallel plates.

The lower plate is stationary and the upper plate moves with uniform velocity. The lower plate is subjected to constant suction (injection) velocity and the upper plate has an equal injection (suction) velocity. The analysis indicates that the velocity of either phase increases as the suction velocity at the lower plate increases. In part II of this chapter, we solve the following problem :

The upper plate initially moves at a uniform velocity while the lower plate is kept stationary. Suddenly, the upper plate decelerates or accelerates according to the law varying exponentially with time. The lower plate is subjected to suction (injection) velocity while there is an equal injection (suction) velocity at the upper plate. The problem is to find the characteristics of the gas-particulate flow in between these two plates under such conditions. Closed form solutions of the governing equations are obtained by the method of separation of variables and analytical expressions for the velocity profiles and skin-friction at the lower plate are obtained. When the upper plate accelerates, skin friction increases non-linearly with time. On the other hand when the upper plate decelerates, skin-friction decreases non-linearly with time. For the time $t \to 0$, the solution for the steady case (discussed in part I) comes out as part of the solution.

## Part I

## Steady gas-particulate flow between porous parallel plates

### 3.2 Mathematical formulation of the problem

We consider the motion of the gas-particulate flow between porous infinite parallel plates. The lower plate at $y = 0$ is at rest and the upper plate at $y = h$ moves with uniform velocity $u_\infty$. The lower plate is subjected to constant suction (injection) velocity $v_w$ and the upper plate has an equal injection (suction) velocity. Equations governing the steady gas-particulate flow between infinite parallel plates are the following :

$$\frac{\partial v}{\partial y} = 0 \tag{3.1}$$

$$\rho v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} - F \rho_p (u - u_p) \tag{3.2}$$

$$\rho v \frac{\partial v}{\partial y} = \mu \frac{\partial^2 v}{\partial y^2} - F \rho_p (v - v_p) \tag{3.3}$$

$$\frac{\partial}{\partial y} (\rho_p \; v_p) = 0 \tag{3.4}$$

$$v_p \frac{\partial u_p}{\partial y} = F(u - u_p) \tag{3.5}$$

$$v_p \frac{\partial v_p}{\partial y} = F(v - v_p) \tag{3.6}$$

Here $(u, v)$ and $(u_p, v_p)$ are the components of velocities of the gas and particle phases along and perpendicular to the plate length respectively, $\mu$ and $\rho$ the viscosity coefficient and density for the gas phase, $\rho_p$ the density for the particle

phase and

$$F = \frac{18\,\mu}{d^2\ \rho_{sp}}$$

where d is the diameter of a particle and $\rho_{sp}$ the material density of the particles.

Boundary conditions

(i) At $y = 0,\ u = 0,\ v = v_p = v_w$ (constant) (3.7)

(ii) At $y = h,\ u = u_p = u_\infty$ (3.8)

We introduce the non-dimensional variables as follows :

$$\left.\begin{aligned} &\bar{y} = \frac{y\sqrt{Re}}{h}\ ,\ \bar{u} = \frac{u}{u_\infty}\ ,\ \bar{u}_p = \frac{u_p}{u_\infty}\ ,\ \bar{v} = \frac{v\sqrt{Re}}{u_\infty} \\ &D = \frac{d}{h}\ ,\ \bar{\rho}_p = \frac{\rho_p}{\rho_{p_o}},\ \bar{\rho}_{sp} = \frac{\rho_{sp}}{\rho_{p_o}}\ ,\ Re = \frac{u_\infty\ \rho h}{\mu} \end{aligned}\right] \qquad (3.9)$$

where $\rho_{p_o}$ is the density of the particulate phase at $y = h$.

Using (3.9), eqs. (3.1) to (3.6) become

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \qquad (3.10)$$

$$\bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F}\ \bar{\rho}_p(\bar{u} - \bar{u}_p) \qquad (3.11)$$

$$\bar{v}\,\frac{\partial \bar{v}}{\partial \bar{y}} = \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} - \bar{F}\ \bar{\rho}_p(\bar{v} - \bar{v}_p) \qquad (3.12)$$

$$\frac{\partial}{\partial \bar{y}}\,(\bar{\rho}_p\ \bar{v}_p) = 0 \qquad (3.13)$$

$$\bar{v}_p \frac{\partial \bar{u}_p}{\partial \bar{y}} = \bar{F}\,\bar{\rho}\,(\bar{u} - \bar{u}_p) \qquad (3.14)$$

$$\bar{v}_p \frac{\partial \bar{v}_p}{\partial \bar{y}} = \bar{F}\,\bar{\rho}\,(\bar{v} - \bar{v}_p) \qquad (3.15)$$

where

$$\bar{F} = \frac{18}{Re\ D^2\ \bar{\rho}_{sp}}$$

Boundary conditions (3.7) and (3.8) become

(i) At $\bar{y} = 0$, $\bar{u} = 0$, $\bar{v} = \bar{v}_p = \bar{v}_w$ (constant) (3.16)

(ii) At $\bar{y} = \bar{h}$, $\bar{u} = \bar{u}_p = 1$ (3.17)

Here

$$\bar{h} = \sqrt{Re}\ .$$

## 3.3 Solution of the governing equations

We solve eqs. (3.10) to (3.15) subject to the boundary conditions (3.16) and (3.17). Using (3.16) and (3.17), eqs. (3.10), (3.12) and (3.13) give

$$\bar{v} = \bar{v}_p = \bar{v}_w \qquad (3.18)$$

$$\bar{\rho}_p = 1 \qquad (3.19)$$

Eq. (3.15) is automatically satisfied.

Using (3.18) and (3.19), eqs. (3.11) and (3.14) become

$$\bar{v}_w \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F}\,(\bar{u} - \bar{u}_p) \qquad (3.20)$$

$$\bar{v}_w \frac{\partial \bar{u}_p}{\partial \bar{y}} = \bar{F}\ \bar{\rho}\ (\bar{u} - \bar{u}_p) \tag{3.21}$$

Eliminating $\bar{u}_p$ from eqs. (3.20) and (3.21), we get

$$\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + \left(\frac{\bar{F}\ \bar{\rho}}{\bar{v}_w} - \bar{v}_w\right) \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F}\ (1 + \bar{\rho}) \frac{\partial \bar{u}}{\partial \bar{y}} = 0 \tag{3.22}$$

Eqs. (3.20) and (3.22) with the boundary conditions (3.16) and (3.17) give the following solution.

$$\bar{u} = \frac{1}{R} \{ S(1 - e^{Q\bar{y}/2} \cosh \frac{P\bar{y}}{2}) + e^{Q\bar{y}/2} \sinh \frac{P\bar{y}}{2}$$

$$\left[ \left[ (1 + \frac{2}{\bar{\rho}})\ \bar{v}_w^2 + \bar{F}\ \bar{\rho}) \right] \coth \frac{P\bar{h}}{2} - P\ \bar{v}_w \right] \} \tag{3.23}$$

$$\bar{u}_p = \frac{1}{R} \{ S + e^{Q\bar{y}/2} \cosh \frac{P\bar{y}}{2} (\bar{v}_w^2 - \bar{F}\ \bar{\rho} + P\ \bar{v}_w \coth \frac{P\bar{h}}{2})$$

$$+ e^{Q\bar{y}/2} \sinh \frac{P\bar{y}}{2} \left[ (\bar{F}\ \bar{\rho} - \bar{v}_w^2) \coth \frac{P\bar{h}}{2} - P\bar{v}_w \right] \} \tag{3.24}$$

where

$$P = \left[ \left(\frac{\bar{F}\ \bar{\rho}}{\bar{v}_w} - \bar{v}_w\right)^2 + 4\bar{F}\ (1 + \bar{\rho}) \right]^{1/2}$$

$$Q = \bar{v}_w - \frac{\bar{F}\ \bar{\rho}}{\bar{v}_w}$$

$$S = \bar{F}\ \bar{\rho} + (1 + \frac{2}{\bar{\rho}})\ \bar{v}_w - P\ \bar{v}_w \coth \frac{P\bar{h}}{2}$$

$$R = S + P\ \bar{v}_w\ e^{Q\bar{h}/2} \operatorname{cosech} \frac{P\bar{h}}{2}$$

Coefficient of skin-friction at the lower plate is

$$C_f = \frac{2}{\sqrt{Re}} \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0} \tag{3.25}$$

where

$$C_f = \frac{\tau}{\frac{1}{2}\rho u_\infty^2}\,,\quad \tau \text{ being the skin-friction coefficient}$$

or $$C_f = \frac{2}{R.\sqrt{Re}} \left[ 2\bar{v}_w^2 \left(1 + \frac{1}{\bar{\rho}}\right)\left(\frac{P}{2} \coth \frac{P\bar{h}}{2} - \bar{v}_w + \frac{\bar{F}\,\bar{\rho}}{\bar{v}_w}\right) \right] \tag{3.26}$$

From eq. (3.24), we get

$$\bar{u}_p(0) = \frac{2}{R}\left(1 + \frac{1}{\bar{\rho}}\right) \bar{v}_w^2 \tag{3.27}$$

Eq. (3.27) shows that particulate velocity at the lower plate depends on the suction (injection) velocity.

For the case of no suction (injection), eqs. (3.10) to (3.15) reduce to the following

$$\bar{u} = \bar{u}_p \tag{3.28}$$

and

$$\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = 0 \tag{3.29}$$

The solution of eq. (3.29) with boundary conditions (3.16) and (3.17) is

$$\bar{u} = \frac{\bar{y}}{\bar{h}} \tag{3.30}$$

The solution (3.30) can also be derived as a particular case from eq. (3.23) when $\bar{v}_w = 0$.

## 3.4 Discussion of the results

We present here some characteristic features of the steady gas-particulate flow between porous infinite plates. For numerical computations, we use the following values of the parameters :

$$\tilde{F} = .8$$

$$\bar{v}_W = -.1, \ -.5$$

$$Re = 1.0$$

Fig. 1 gives the variation of the velocities of the gas phase and the particle phase with changes in the suction velocity. We find that as the suction velocity increases, velocity of either phase increases. Also, the velocity of the particle phase at the lower plate increases with increase in the suction velocity.

# Part II

## Unsteady flow between porous infinite parallel plates

## 3.5 Mathematical formulation of the problem

We consider the unsteady flow between porous infinite parallel plates. The upper plate initially moves at a uniform velocity while the lower plate is kept stationary. Suddenly, the upper plate decelerates or accelerates according to the law varying exponentially with time. The lower plate is subjected to suction (injection) velocity while there is an equal injection (suction) velocity at the upper plate. Equations governing the

unsteady flow between porous infinite plates are :

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3.31}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F} \bar{\rho}_p (\bar{u} - \bar{u}_p) \tag{3.32}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} = \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} - \bar{F} \bar{\rho}_p (\bar{v} - \bar{v}_p) \tag{3.33}$$

$$\frac{\partial \bar{\rho}_p}{\partial \bar{t}} + \frac{\partial}{\partial \bar{y}} (\bar{\rho}_p \bar{v}_p) = 0 \tag{3.34}$$

$$\frac{\partial \bar{u}_p}{\partial \bar{t}} + \bar{v}_p \frac{\partial \bar{u}_p}{\partial \bar{y}} = \bar{F} \bar{\rho} (\bar{u} - \bar{u}_p) \tag{3.35}$$

$$\frac{\partial \bar{v}_p}{\partial \bar{t}} + \bar{v}_p \frac{\partial \bar{v}_p}{\partial \bar{y}} = \bar{F} \bar{\rho} (\bar{v} - \bar{v}_p) \tag{3.36}$$

Here

$$\bar{t} = \frac{t \, u_\infty}{h}$$

and the rest of the variables have same meaning as defined in (3.9).

Boundary conditions are the following :

(i) At $\bar{y} = 0$, $\bar{u} = 0$, $\bar{v} = \bar{v}_p = \bar{v}_w$ (constant) (3.37)

(ii) At $\bar{y} = \bar{h}$, $\bar{u} = \bar{u}_p = e^{\alpha \bar{t}}$, $\bar{\rho}_p = 1$ (3.38)

where $\alpha$ is a non-dimensional constant number. $\alpha$ is positive for accelerating upper plate and negative for a decelerating upper plate.

## 3.6 Solution of the problem

Eqs. (3.31), (3.33) and (3.34) subject to the boundary conditions (3.37) and (3.38) give

$$\bar{v} = \bar{v}_p = \bar{v}_w \tag{3.39}$$

$$\bar{\rho}_p = 1 \tag{3.40}$$

and then eq. (3.36) is automatically satisfied.

With eqs. (3.39) and (3.40), eqs. (3.32) and (3.35) become

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}_w \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{F}(\bar{u} - \bar{u}_p) \tag{3.41}$$

$$\frac{\partial \bar{u}_p}{\partial \bar{t}} + \bar{v}_w \frac{\partial \bar{u}_p}{\partial \bar{y}} = \bar{F}\,\bar{\rho}\,(\bar{u} - \bar{u}_p) \tag{3.42}$$

Let

$$\bar{u} = e^{\alpha \bar{t}}\, G(\bar{y}) \tag{3.43}$$

$$\bar{u}_p = e^{\alpha \bar{t}}\, H(\bar{y}) \tag{3.44}$$

where

$$G(\bar{h}) = H(\bar{h}) = 1 \tag{3.45}$$

$$G(0) = 0 \tag{3.46}$$

With eqs. (3.43) and (3.44), eqs. (3.41) and (3.42) become :

$$G'' - \bar{v}_w G' - (\bar{F} + \alpha)\, G + \bar{F}\, H = 0 \tag{3.47}$$

$$\bar{v}_w H' + (\bar{F}\,\bar{\rho} + \alpha)\, H - \bar{F}\,\bar{\rho}\, G = 0 \tag{3.48}$$

where ' denotes the derivative with respect to $\bar{y}$.

Eliminating H from eqs. (3.47) and (3.48), we get

$$(D^3 + a\, D^2 + b\, D + c)\, G = 0 \tag{3.49}$$

where

$$D \equiv \frac{\partial}{\partial y}$$

$$a = \frac{1}{\bar{v}_w}\left(\bar{F}\,\bar{\rho} + \alpha - \bar{v}_w^2\right)$$

$$b = -(\bar{F} + \bar{F}\,\bar{\rho} + 2\alpha)$$

$$c = \frac{1}{\bar{v}_w}\left[\bar{F}^2\,\bar{\rho} - (\bar{F}\,\bar{\rho} + \alpha)\,(\bar{F} + \alpha)\right]$$

Let

$$D = m - \frac{a}{3} \tag{3.50}$$

Using (3.50), operator with in round bracket in eq. (3.49) is transformed to

$$m^3 + pm + q$$

where

$$p = b - \frac{a^2}{3}$$

$$q = c - \frac{ba}{3} + \frac{2a^3}{27}$$

The discriminant $\Delta$ of the equation

$$m^3 + pm + q = 0 \tag{3.51}$$

is given by

$$\Delta = 4p^3 + 27q^2$$

or $$\Delta = 4b^3 + 27c^2 + 4a^3c - a^2b^2 - 18abc \tag{3.52}$$

Depending on the values of the parameters, $\Delta$ may be positive or negative.

Analytically, it is difficult to say about the sign of $\Delta$ with varying parameters. Numerical computations have shown that positive values of $\alpha$ determine $\Delta$ as negative while negative

values of $\alpha$ give $\Delta$ as positive. This means that for accelerating upper plate, $\Delta$ is negative and for a decelerating upper plate, $\Delta$ is positive.

Case I

Let $\alpha$ be positive, that is, upper plate is moving with a velocity increasing exponentially with time.

Here, $\Delta$ is negative and then eq. (3.51) has all the three real roots.

Let $\beta$, $\gamma$, $\delta$ be the three real roots, then

$$\beta = 2\sqrt{-p/3}\cos\frac{\phi}{3}$$

$$\gamma = 2\sqrt{-p/3}\cos\left(60^\circ - \frac{\phi}{3}\right) = \lambda + \mu_1$$

$$\delta = 2\sqrt{-p/3}\cos\left(60^\circ + \frac{\phi}{3}\right) = \lambda - \mu_1$$

where

$$\phi = \tan^{-1}\left[\frac{27q^2 + 4p^3}{54q}\right]$$

Corresponding roots of the equation

$$D^3 + aD^2 + bD + c = 0$$

then become

$$\beta_1 = 2\sqrt{-p/3}\cos\frac{\phi}{3} - \frac{a}{3}$$

$$\gamma_1 = \lambda - \frac{a}{3} + \mu_1 = \lambda_1 + \mu_1$$

$$\delta_1 = \lambda - \frac{a}{3} - \mu_1 = \lambda_1 - \mu_1$$

where

$$\lambda_1 = \lambda - \frac{a}{3}$$

Hence solution of eqs. (3.47) and (3.49) using boundary conditions (3.45) and (3.46) is

$$G = \frac{\sinh \mu_1 \bar{y}}{\sinh \mu_1 \bar{h}} e^{\lambda_1(\bar{y}-\bar{h})} + \frac{A}{B} \{ e^{\beta_1 \bar{y}} - \frac{\sinh \mu_1 \bar{y}}{\sinh \mu_1 \bar{h}} e^{\beta_1 \bar{h}} e^{\lambda_1(\bar{y}-\bar{h})} - \frac{\sinh \mu_1(\bar{h}-\bar{y})}{\sinh \mu_1 \bar{h}} e^{\lambda_1 \bar{y}} \} \qquad (3.53)$$

and

$$H = \frac{e^{\lambda_1(\bar{y}-\bar{h})}}{\sinh \mu_1 \bar{h}} (P_1 \sinh \mu_1 \bar{y} + Q_1 \cosh \mu_1 \bar{y}) + \frac{A}{B} \{ ( \frac{\bar{F}+\alpha}{\bar{F}} + \frac{\bar{v}_w \beta_1}{\bar{F}} - \frac{\beta_1^2}{\bar{F}} ) e^{\beta_1 \bar{y}} - \frac{e^{\beta_1 \bar{h}} e^{\lambda_1 \bar{y}}}{e^{\lambda_1 \bar{h}} \sinh \mu_1 \bar{h}} (P_1 \sinh \mu_1 \bar{y} + Q_1 \cosh \mu_1 \bar{y}) - \frac{e^{\lambda_1 \bar{y}}}{\sinh \mu_1 \bar{h}} \left[ P_1 \sinh \mu_1(\bar{h}-\bar{y}) - Q_1 \cosh \mu_1(\bar{h}-\bar{y}) \right] \qquad (3.54)$$

where

$$P_1 = \frac{\bar{F}+\alpha}{\bar{F}} + \frac{\bar{v}_w \lambda_1}{\bar{F}} - \frac{\lambda_1^2 + \mu_1^2}{\bar{F}}$$

$$Q_1 = \frac{\bar{v}_w \mu_1}{\bar{F}} - \frac{2 \mu_1 \lambda_1}{\bar{F}}$$

$$A = \bar{F}(1-P_1) \sinh \mu_1 \bar{h} - \bar{F} Q_1 \cosh \mu_1 \bar{h}$$

and

$$B = \left[ \bar{F}(1-P_1) + \beta(\bar{v}_w - \beta_1) - \alpha \right] e^{\beta_1 \bar{h}} \sinh \mu_1 \bar{h} - Q_1 \bar{F} e^{\beta_1 \bar{h}} \cosh \mu_1 \bar{h} + Q_1 \bar{F} e^{\lambda_1 \bar{h}}$$

Eqs. (3.43) and (3.44), using eqs. (3.53) and (3.54) give $\bar{u}$ and $\bar{u}_p$ respectively.

At the initial time $t = 0$, the solution given by eqs.(3.53) and (3.54) gives the steady state solution when the upper plate moves with the uniform velocity $u_\infty$. This is exactly the same as given by eqs. (3.23) and (3.24) in Part I.

Eqs. (3.25) and (3.53) give

$$C_f = \frac{2}{\sqrt{Re}} \frac{e^{\alpha\bar{t}} e^{\lambda_1 \bar{h}}}{\sinh \mu_1 \bar{h}}$$

$$+ \frac{2Ae^{\alpha\bar{t}}}{B\sqrt{Re}} \{ \beta_1 - \frac{1}{\sinh \mu_1 \bar{h}} \Big[ \mu_1 \; e^{(\beta_1 - \lambda_1)\bar{h}}$$

$$+ \lambda_1 \sinh \mu_1 \bar{h} - \mu_1 \cosh \mu_1 \bar{h} \Big] \} \tag{3.55}$$

Case II

Let $\alpha$ be negative so that the upper plate is moving with a velocity decreasing exponentially with time. Here $\Delta$ is positive and then eq. (3.51) has one real root and two complex conjugate roots.

Let $\beta_2'$ be the real root and $\gamma_2'$, $\delta_2'$ the complex conjugate roots

$$\beta_2' = u_1 + v_1$$

$$\gamma_2' = - \frac{u_1 + v_1}{2} + \frac{i\sqrt{3}\,(u_1 - v_1)}{2}$$

$$\delta_2' = - \frac{u_1 + v_1}{2} - \frac{i\sqrt{3}\,(u_1 - v_1)}{2}$$

where

$$u_1 = \left[-\left(\frac{c}{2} - \frac{ab}{6} + \frac{a^3}{27}\right) + \left(\frac{4b^3+27c^2+4a^3c-a^2b^2-18abc}{108}\right)^{1/2}\right]^{1/3}$$

$$v_1 = \left[-\left(\frac{c}{2} - \frac{ab}{6} + \frac{a^3}{27}\right) - \left(\frac{4b^3+27c^2+4a^3c-a^2b^2-18abc}{108}\right)^{1/2}\right]^{1/3}$$

Corresponding roots of

$$D^3 + aD^2 + bD = c = 0$$

are

$$\beta_2 = u_1 + v_1 - \frac{a}{3}$$

$$\gamma_2 = \left\{-\frac{u_1 + v_1}{2} - \frac{a}{3}\right\} + i\left\{\frac{\sqrt{3}\,(u_1 - v_1)}{2}\right\} = \lambda_2 + i\,\mu_2$$

$$\delta_2 = \left\{-\frac{u_1 + v_1}{2} - \frac{a}{3}\right\} - i\left\{\frac{\sqrt{3}\,(u_1 - v_1)}{2}\right\} = \lambda_2 - i\,\mu_2$$

The solution of eqs. (3.47) and (3.49) using boundary conditions (3.45) and (3.46) is

$$G = \frac{\sin \mu_2 \bar{y}}{\sin \mu_2 \bar{h}}\, e^{-\lambda_2(\bar{h}-\bar{y})} + \frac{A_2}{B_2}\left\{e^{\beta_2 \bar{y}} - \frac{\sin \mu_2 \bar{y}}{\sin \mu_2 \bar{h}}\, e^{\beta_2 \bar{h}}\, e^{\lambda_2(\bar{y}-\bar{h})} - \frac{\sin \mu_2(\bar{h}-\bar{y})}{\sin \mu_2 \bar{h}}\, e^{\lambda_2 \bar{y}}\right\} \qquad (3.56)$$

and

$$H = \frac{e^{\lambda_2(\bar{y}-\bar{h})}}{\sin \mu_2 \bar{h}}\,(P_2 \sin \mu_2 \bar{y} + Q_2 \cos \mu_2 \bar{y}) + \frac{A_2}{B_2}\left\{\frac{1}{\bar{F}}\,(\bar{F} - \alpha + \bar{v}_o\,\alpha - \alpha^2)\, e^{\beta_2 \bar{y}}\right.$$

$$- \frac{e^{\beta_2 \bar{h}} \; e^{\lambda_2(\bar{y}-\bar{h})}}{\sin \mu_2 \bar{h}} (P_2 \sin \mu_2 \bar{y} + Q_2 \cos \mu_2 \bar{y})$$

$$- \frac{e^{\lambda_2 \bar{y}}}{\sin \mu_2 \bar{h}} (P_2 \sin \mu_2(\bar{h}-\bar{y}) - Q_2 \cos \mu_2(\bar{h}-\bar{y}))\} \qquad (3.57)$$

Here

$$P_2 = \frac{\bar{F}-\alpha}{\bar{F}} + \frac{\bar{v}_w \lambda_2}{\bar{F}} - \frac{\lambda_2^2 - \mu_2^2}{\bar{F}}$$

$$Q_2 = \frac{\bar{v}_w \mu_2}{\bar{F}} - \frac{2\lambda_2 \mu_2}{\bar{F}}$$

$$A_2 = \bar{F}\,(1-P_2) \sin \mu_2 \bar{h} - Q_2 \bar{F} \cos \mu_2 \bar{h}$$

$$B_2 = (\bar{F}(1-P_2) + \beta_2(\bar{v}_w-\beta_2) - \alpha)\, e^{\beta_2 \bar{h}} \sin \mu_2 \bar{h}$$

$$- Q_2 \bar{F} e^{\beta_2 \bar{h}} \cos \mu_2 \bar{h} + Q_2 \bar{F} e^{\lambda_2 \bar{h}}$$

Eqs. (3.56) and (3.57), using eqs. (3.43) and (3.44) give respectively the expressions for $\bar{u}$ and $\bar{u}_p$. We further note that as part of this solution, we get steady state solution.

Eq. (3.25), using eq. (3.56) gives

$$C_f = \frac{2e^{\alpha \bar{t}} \; e^{-\lambda_2 \bar{h}}}{\sqrt{Re}\, \sin \mu_2 \bar{h}}$$

$$+ \frac{2A_2 \, e^{\alpha \bar{t}}}{B_2 \sqrt{Re}} \{ \beta_2 - \frac{1}{\sin \mu_2 \bar{h}} (\mu_2 \, e^{(\beta_2-\lambda_2)\bar{h}}$$

$$+ \lambda_2 \sin \mu_2 \bar{h} - \mu_2 \cos \mu_2 \bar{h})\} \qquad (3.58)$$

## 3.7 Discussion of the results

In this section, we present some characteristic features of unsteady flow between parallel plates for both the cases.

For numerical computations, following values of the parameters are considered.

$$\bar{F} = .8$$

$$\bar{v}_0 = -.1, \ -.5$$

$$Re = 1.0$$

Fig. 2 gives the velocity distributions of gas and particulate phases at different times when the upper plate moves with velocity increasing with time. This graph shows that as the time increases, the difference in velocities of the gas and particle phases increases. Fig. 3 gives the velocity profiles for the gas and particulate phases at different times when the upper plate moves with velocity decreasing with time. This figure indicates that as the time increases, the difference in velocities of both the phases decreases.

Fig. 6 gives the variation of skin-friction at the lower plate with time for both the cases. This figure indicates that in case the upper plate moves with velocity increasing with time, skin-friction increases with time. When the upper plate moves with velocity decreasing with time, skin-friction decreases with time.

STEADY CASE

$\bar{F} = 0{\cdot}8,\ \bar{h} = 1{\cdot}0,\ \bar{t} = 0{\cdot}0.$

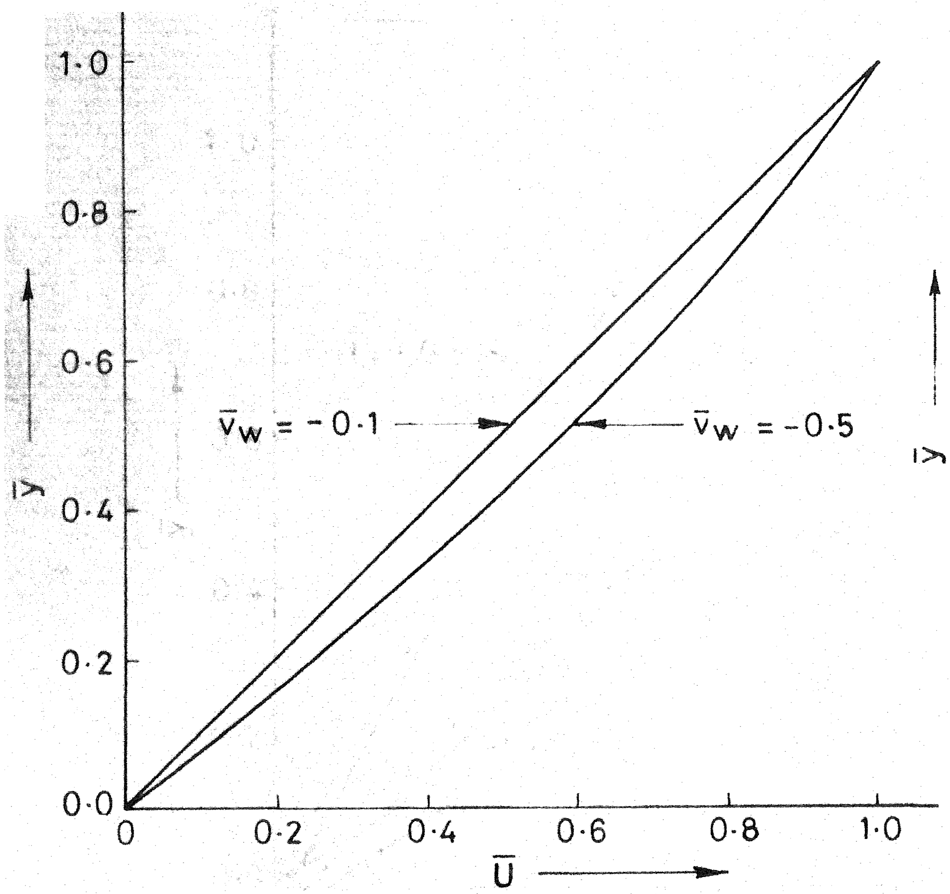



FIG. 1a. VARIATION OF $\bar{U}$ WITH SUCTION VELOCITY $\bar{V}_W$.

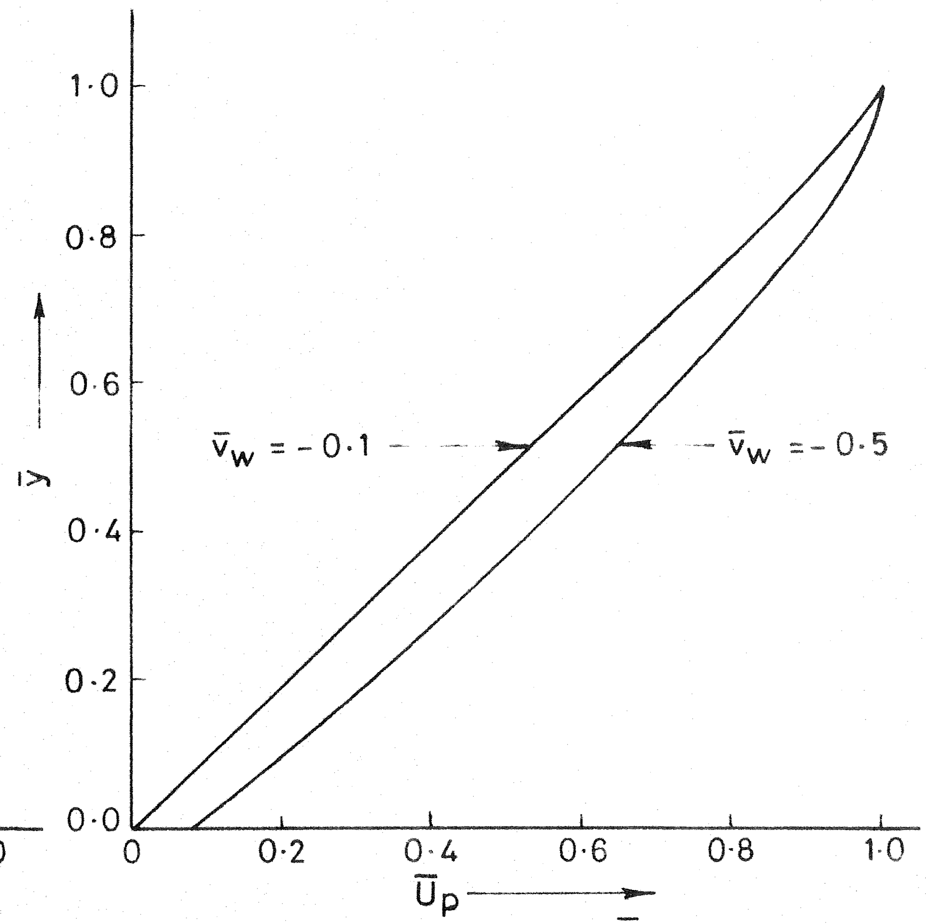



FIG. 1b. VARIATION OF $\bar{U}_p$ WITH SUCT VELOCITY $\bar{V}_W$.

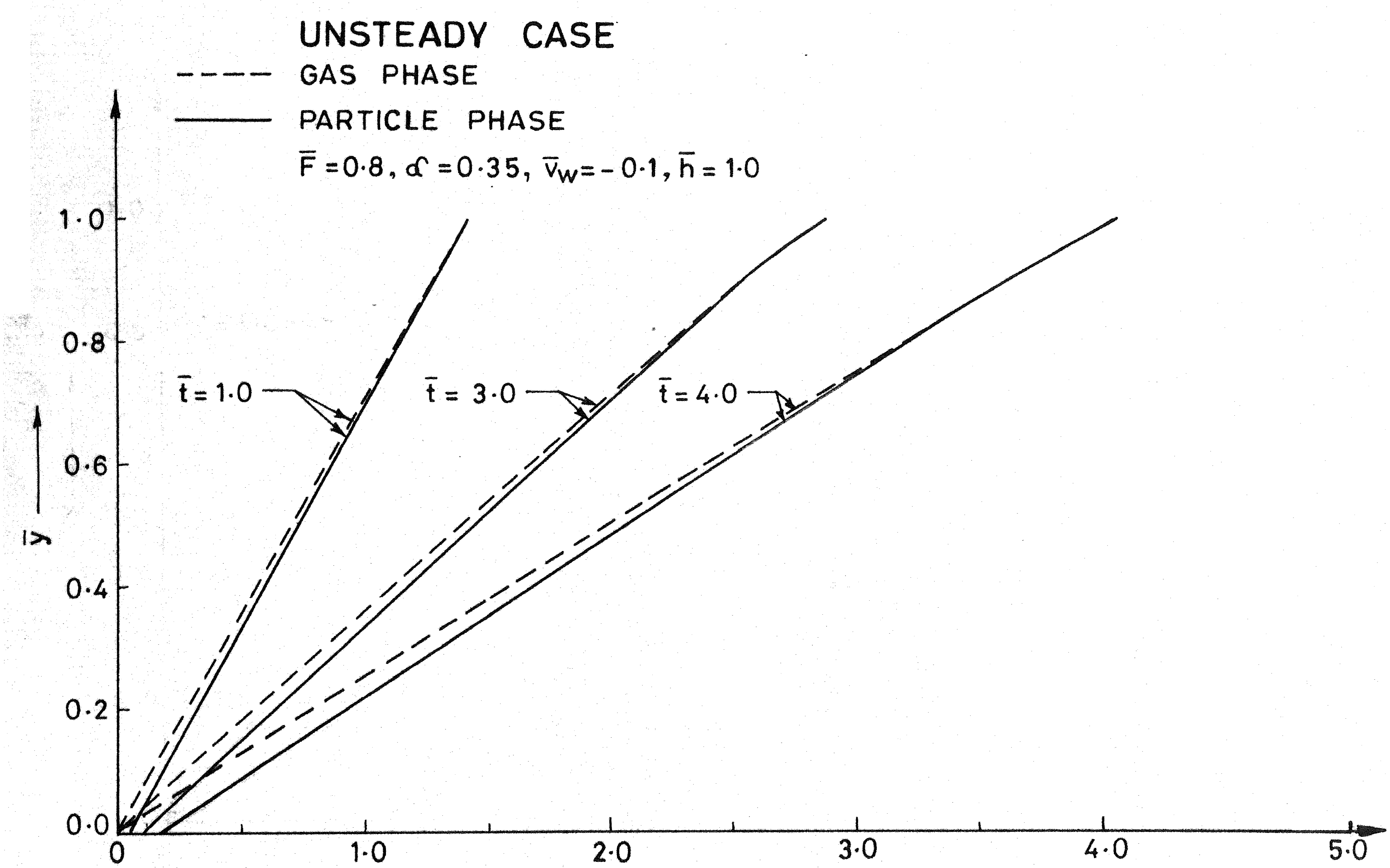



FIG. 2 VARIATION OF $\bar{U}$, $\bar{U}_p$ WITH UPPER PLATE MOVING WITH VELOCITY INCREASING WITH TIME.

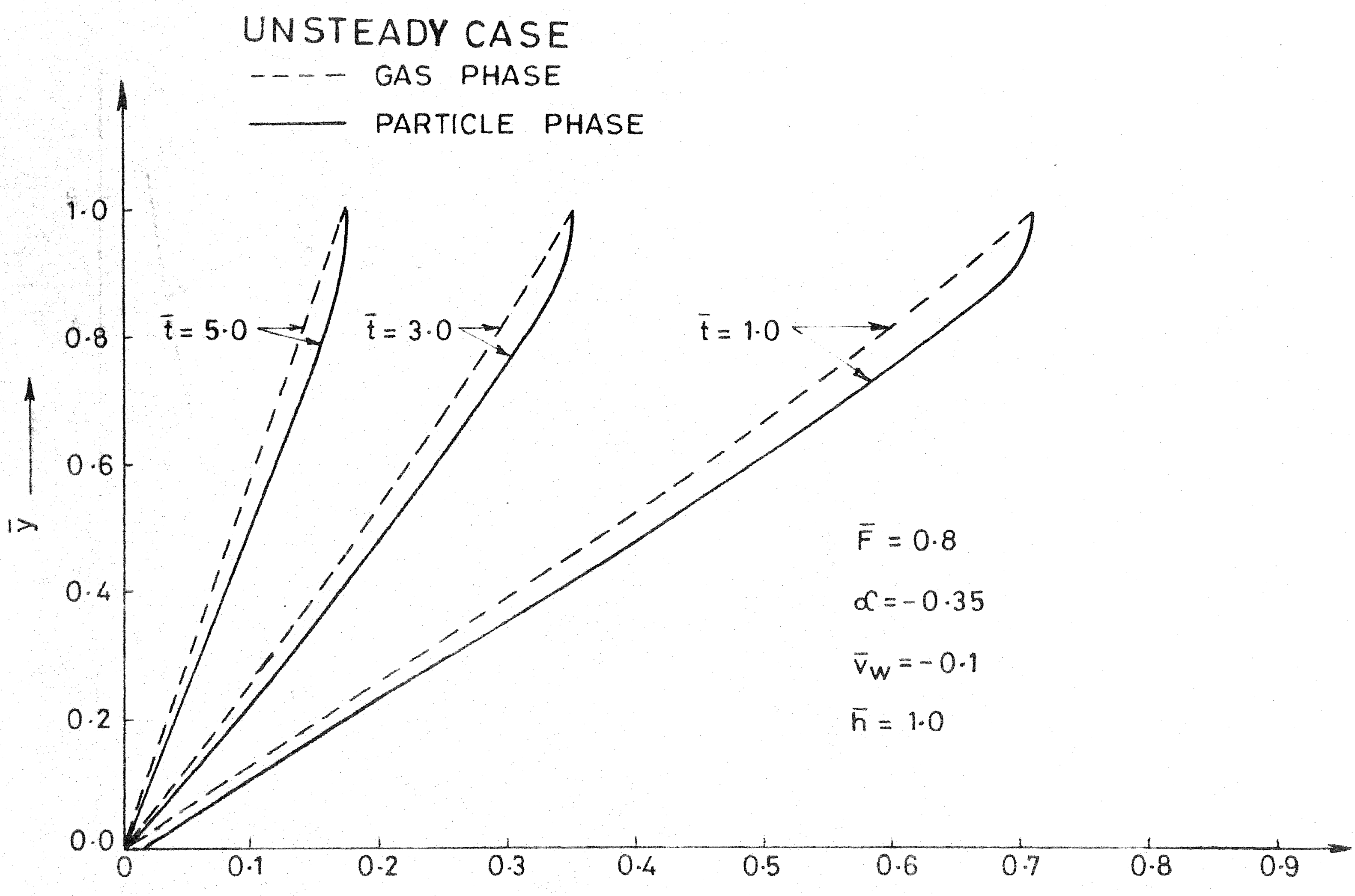



FIG.3 VARIATION OF $\bar{U}$ AND $\bar{U}_p$ WITH UPPER PLATE MOVING WITH VELOCITY DECREASING EXPONENTIALLY WITH TIME.

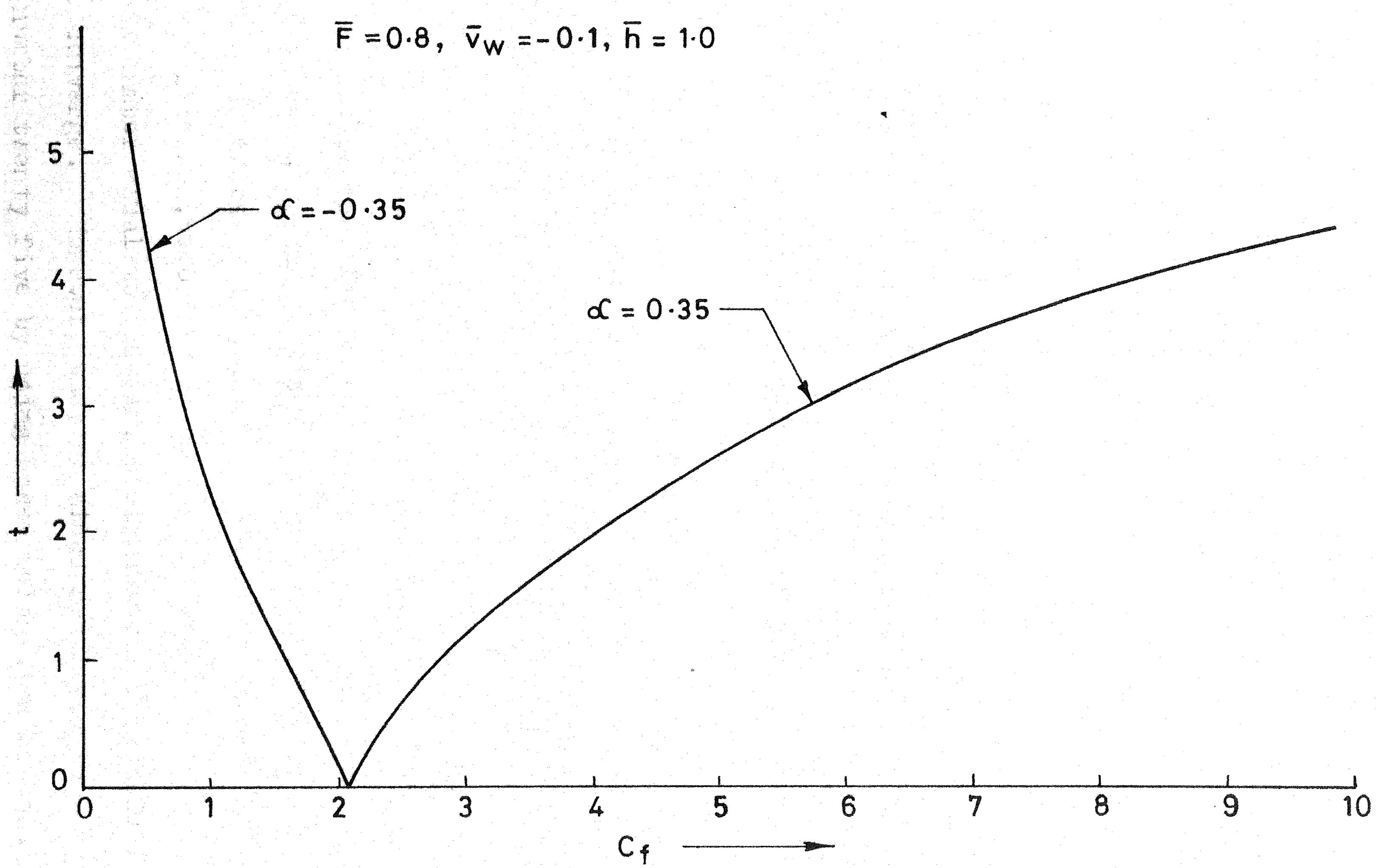



FIG. 4. VARIATION OF $C_f$ WITH TIME.

# CHAPTER IV

## GAS-PARTICULATE FLOW THROUGH TUBES OF VARYING CROSS-SECTION

### 4.1 Introduction

Blasius (1910) analyzed the motion of an incompressible fluid through a symmetrical capillary with constriction which starts and ends with the same diameter. He developed an asymptotic series solution and obtained analytical solutions for the zeroth and the first order terms. His analysis predicted no extra pressure losses (due to varying cross-section) above the pressure losses predicated by Stokes in straight pipes. Tanner (1966) extended the perturbation analysis of Blasius (1910) and obtained closed form expression for the pressure loss in a class of viscometers. Lee and Fung (1970) investigated the flow in a circular tube of bell shaped constriction specified mathematically by a Gaussian distribution curve. Using finite difference technique, they obtained numerical solutions of equations for stream lines, shear stresses, velocity components and pressure losses. At Reynolds number equal to fifteen, their numerical procedure failed to converge. However, they obtained results upto Reynolds number twenty five by using under relaxation method. They found the presence of an eddy on the downstream side of the constriction when the Reynolds number exceeded 9.9.

Manton (1971) considered the second order terms neglected by Tanner (1966) in the momentum equations. He developed an asymptotic series solution in the ascending powers of $\epsilon$, a small positive number which gave the variation of the cross-section of the tube with the axial distance. His results for shear stress at the wall agreed qualitatively with the numerical results of Lee and Fung (1970). Lee and Fung (1970) and Manton (1971) indicated that the point of maximum shear stress occured at the upstream side of the constriction. Chow and Soda (1972) considered the flow in a tube with continuous constriction or obstruction. They developed the series solution in terms of the ascending powers of $\delta$, the ratio of the mean radius of the tube to the characteristic length of the tube and obtained closed form solutions for first three terms of the series. Their results were valid for a case when the spread of roughness was large in comparison with the mean radius of the tube. In their analysis, they discussed seperation and reattachment points for conduits and sinusoidal wall variation. Recently, Doorean (1978) and Macdonald (1978) solved the problem in circular tube with exponentially varying cross-section of the walls. Macdonald (1978) used Galerkin's Kantrovich scheme and Doorean (1978) developed a series solution by transforming the domain of the tube into a rectangular domain.

Attention had also been given to gas-particulate flows through tubes of uniform and varying cross-sections. Karnis,

Goldsmith and Mason (1966) found experimentally that in the flow with suspended particles through a tube of uniform cross-section, the gas-phase velocity and the particle phase velocity were identical. Later, Karnis, Goldsmith and Mason (1966) observed that for large volume fraction of the particles, there was a thin layer near the wall where the particle density was almost negligible, that is, there developed a particle free region near the wall. Sproull (1961) observed experimentally that by adding dust particles in the turbulent flow through a pipe of uniform cross-section, shear stress at the wall decreased. Saffman (1962) carried out the stability analysis for dusty gases. His analysis showed that the addition of fine dust particles destabilized the gas flow, whereas the addition of coarse dust particles stabilized the gas flow.

Kaimal (1977) considered the gas-particulate flow through a tube with local constriction specified mathematically by Gaussian distribution curve. He developed an asymptotic series solution in terms of $\epsilon$, giving the variation of the cross-section of the tube and calculated first three terms. His results were valid for constant gas and particulate densities. He obtained identical zeroth order solutions for the gas and particle phases. Even at Reynolds number of unity he obtained eddy-like motion. Moreover, his analysis indicated that by adding coarse dust particles to the gas flow,

the shear stress at the wall was increased. This was in contradiction to the work of Sproull (1961). Mathematically, he solved an overdeterminate system of governing equations, which probably caused such discripances as stated above.

In the present analysis, we reconsider the problem solved by Kaimal (1977) and remove some of the short-comings in the analysis as stated in the above paragraph. We consider variable density of the particle phase and develop an asymptotic series solution in terms of small parameter $\epsilon$, which gives the variation of the cross-section of the tube. We obtain the analytical expressions for the first three terms of the series. The present analysis indicates that the particles are migrated axially and radially towards the wall, thereby resulting in excessive accumulation at the wall. Our three term series solution does not predict the formation of eddy upto Reynolds number equal to eighty. We further observe that by adding coarse dust, the shear stress at the wall is decreased, which agrees qualitatively with the result of Sproull (1961).

## 4.2 Formulation of the problem

Consider the steady gas-particulate flow through an axisymmetric tube of varying cross-section. We select the cylindrical polar coordinates $(R,\theta,X)$, where $R = 0$ is the line of symmetry and $X = 0$ gives the minimum characteristic

radius $a_o$. The radius of the tube varies slowly with the axial distance 'X' viz.

$$R = a(X) = a_o S(\epsilon X/a_o), \; 0 < \epsilon < 1 \tag{4.1}$$

Let $(U,V)$ and $(U_p,V_p)$ denote the velocity components of the fluid and the particle phases in the axial and the radial directions respectively. The governing equations for steady motion of a gas-particulate system are

$$UU_X + VU_R + \frac{P_X}{\rho} = \nu(U_{XX} + U_{RR} + \frac{1}{R}U_R) - \frac{F\rho_p}{\rho}(U-U_p) \tag{4.2}$$

$$UV_X + VV_R + \frac{P_R}{\rho} = \nu(V_{XX} + V_{RR} + \frac{1}{R}V_R - \frac{1}{R^2}V) - \frac{F\rho_p}{\rho}(V-V_p) \tag{4.3}$$

$$(\rho U)_X + \frac{1}{R}(\rho V) + (\rho V)_R = 0 \tag{4.4}$$

$$U_p U_{pX} + V_p U_{pR} = F(U-U_p) \tag{4.5}$$

$$U_p V_{pX} + V_p V_{pR} = F(V-V_p) \tag{4.6}$$

$$(\rho_p U_p)_X + \frac{1}{R}(\rho_p V_p) + (\rho_p V_p)_R = 0 \tag{4.7}$$

where $\rho$, $P$, $\nu$ are the density, pressure and kinematic viscosity respectively for the gas phase; $\rho_p$ the density of the particulate cloud and $F$ the interaction parameter given by

$$F = \frac{18\mu}{d^2 \rho_{sp}}$$

Here, $\mu$ is the viscosity coefficient for the gas phase; $d$ the diameter of a particle and $\rho_{sp}$ the material density of the particle phase. In eqs. (4.2) to (4.7), suffixes X and R

denote the partial derivatives of the dependent variables with respect to X and R respectively.

Boundary conditions

I For the fluid phase

(i) No slip condition at the wall gives

$$U + a_X V = 0 \tag{4.8}$$

(ii) No flow condition across the wall gives

$$V - a_X U = 0 \tag{4.9}$$

(iii) Due to symmetry of the flow at the axis of the tube,

$$V = 0 \text{ and } \frac{\partial U}{\partial R} = 0 \quad \text{at} \quad R = 0 \tag{4.10}$$

II For the particle phase

(i) No flow across the wall gives

$$V_p - a_X U_p = 0 \tag{4.11}$$

(ii) Due to symmetry of the flow at the axis of the tube,

$$V_p = 0 \text{ and } \frac{\partial U_p}{\partial R} = 0 \text{ at } R = 0 \tag{4.12a, b}$$

It is to be noted that we have not used the no slip condition at the wall for the particle phase.

## 4.3 Method of solution

Let us define the stream functions $\psi$ and $\psi_p$ for the gas and particle phases respectively as follows:

$$U = \frac{1}{R} \psi_R , \quad V = -\frac{1}{R} \psi_X \tag{4.13}$$

$$\rho_p U_p = \frac{1}{R}\ \psi_{pR},\ \rho_p V_p = -\frac{1}{R}\ \psi_{pX} \tag{4.14}$$

With eqs. (4.13) and (4.14), eqs. (4.4) and (4.7) are automatically satisfied. Also, with eqs. (4.13) and (4.14), eqs. (4.2), (4.3), (4.5) and (4.6) become

$$\frac{1}{R^2}\psi_R\psi_{RX} + \frac{1}{R^3}\psi_X\psi_{XR} - \frac{1}{R^2}\psi_X\psi_{RR} + \frac{P_X}{\rho}$$
$$= \nu\left[\frac{1}{R}(\psi_{RXX}+\psi_{RRR}) - \frac{1}{R^2}\psi_{RR} + \frac{1}{R^3}\psi_R\right]$$
$$-\frac{F\rho_p}{\rho_R}\psi_R + \frac{F}{\rho_R}\psi_{PR} \tag{4.15}$$

$$\frac{1}{R^2}\psi_R\psi_{XX} - \frac{1}{R^2}\psi_X\psi_{XR} + \frac{1}{R^3}\psi_X^2 - \frac{P_R}{\rho}$$
$$= \nu\left[\frac{1}{R}(\psi_{XXX}+\psi_{XRR}) - \frac{1}{R^2}\psi_{XR}\right]$$
$$-\frac{F\rho_p}{\rho_R}\psi_X + \frac{F}{\rho R}\psi_{pX} \tag{4.16}$$

$$\frac{1}{R}\psi_{pR}(\rho_p\psi_{pRX} - \rho_{pX}\psi_{PR}) + \frac{1}{R}\psi_{pX}(\frac{\rho_p}{R}\psi_{pR} + \rho_{pR}\psi_{pR}$$
$$-\rho_p\psi_{pRR}) = F\rho_p^3\psi_R - F\rho_p^2\psi_{PR} \tag{4.17}$$

$$\frac{1}{R}\psi_{pR}(\rho_{pX}\psi_{pX} - \rho_p\psi_{pXX}) - \frac{1}{R}\psi_{pX}(\frac{\rho_p}{R}\psi_{pX} + \rho_{pR}\psi_{pX}$$
$$-\rho_p\psi_{pRX}) = -F\rho_p^3\psi_R + F\rho_p^2\psi_{pX} \tag{4.18}$$

Boundary conditions become

(i) At $R = a(X)$,

$$\psi = \psi_o,\quad \psi_R = 0 \quad\text{and}\quad \psi_p = \psi_{p_o} \tag{4.19a,b,c}$$

(ii) At $R = 0$,

$$\psi = 0, \ \frac{1}{R}\psi_X = 0, \ \left(\frac{1}{R}\psi_R\right)_R = 0 \qquad (4.20a,b,c)$$

$$\psi_p = 0, \ \frac{1}{R}\psi_{pX} = 0, \ \left(\frac{1}{R}\frac{1}{\rho_p}\psi_{pR}\right)_R = 0 \qquad (4.21a,b,c)$$

From eq. (4.1), we notice that when $\epsilon \to 0$, the tube is of constant radius $a_o$ provided $S(0) = 1$. For a non-zero $\epsilon$, the variation of $S$ in the axial direction depends on $\epsilon X$ instead of $X$ alone.

We nondimensionalize the variables as follows:

$$\left.\begin{aligned} &\xi = \frac{\epsilon X}{a_o}, \ \eta = \frac{R}{a_o}, \ \phi(\xi,\eta) = \psi/\psi_o, \ \phi_p(\xi,\eta) = \psi_p/\psi_{p_o} \\ &\bar{\rho}_p = \rho_p/\rho_{p_o}, \ \bar{\rho} = \frac{\rho}{\rho_{p_o}}, \ q = \frac{Pa_o^3}{\rho\nu\psi_o} \end{aligned}\right] \qquad (4.22)$$

$\rho_{p_o}$ being the constant density of the particle phase.

Using (4.22) in eqs. (4.15) to (4.18), we get the governing equations of motion in the nondimensional form as follows:

$$R_e\left(\frac{\epsilon}{\eta^2}\phi_\eta\phi_{\eta\xi} - \frac{\epsilon}{\eta^2}\phi_\xi\phi_{\eta\eta} + \frac{\epsilon}{\eta^3}\phi_\xi\phi_\eta\right) + \epsilon q_\xi$$
$$= \frac{\epsilon^2}{\eta}\phi_{\eta\xi\xi} + \frac{1}{\eta}\phi_{\eta\eta\eta} - \frac{1}{\eta^2}\phi_{\eta\eta} + \frac{1}{\eta^3}\phi_\eta - \frac{\alpha}{\eta}\bar{\rho}_p\phi_\eta + \frac{\beta}{\eta}\phi_{p\eta} \qquad (4.23)$$

$$R_e\left(\frac{\epsilon^2}{\eta^2}\phi_\eta\phi_{\xi\xi} - \frac{\epsilon^2}{\eta^2}\phi_\xi\phi_{\xi\eta} + \frac{\epsilon^2}{\eta^3}\phi_\xi^2\right) - q_\eta$$
$$= \frac{\epsilon^3}{\eta}\phi_{\xi\xi\xi} + \frac{\epsilon}{\eta}\phi_{\xi\eta\eta} - \frac{\epsilon}{\eta^2}\phi_{\xi\eta} - \epsilon\frac{\alpha}{\eta}\bar{\rho}_p\phi_\xi + \epsilon\frac{\beta}{\eta}\phi_{p\xi} \qquad (4.24)$$

$$\frac{\epsilon}{\eta}\,\phi_{p\eta}\,\phi_{p\eta\xi}\,\bar{\rho}_p - \frac{\epsilon}{\eta}\,\phi_{p\eta}^2\,\bar{\rho}_{p\xi} + \frac{\epsilon}{\eta}\,\phi_{p\xi}\,\phi_{p\eta}\,\bar{\rho}_{p\eta} + \frac{\epsilon}{\eta^2}\,\phi_{p\eta}\,\phi_{p\xi}\,\bar{\rho}_p$$

$$-\frac{\epsilon}{\eta}\,\phi_{p\xi}\,\phi_{p\eta\eta}\,\bar{\rho}_p = \bar{\alpha}\,\bar{\rho}_p^{3}\,\phi_\eta - \bar{\beta}\,\bar{\rho}_p^2\,\phi_{p\eta} \tag{4.25}$$

$$\frac{\epsilon^2}{\eta}\,\phi_{p\eta}\,\phi_{p\xi}\,\bar{\rho}_{p\xi} - \frac{\epsilon^2}{\eta}\,\phi_{p\xi}^2\,\bar{\rho}_{p\eta} - \frac{\epsilon^2}{\eta}\,\phi_{p\eta}\,\phi_{p\xi\xi}\,\bar{\rho}_p - \frac{\epsilon^2}{\eta^2}\,\phi_{p\xi}^2\,\bar{\rho}_p$$

$$= \frac{\epsilon^2}{\eta}\,\bar{\rho}_p\,\phi_{p\xi\eta}\,\phi_{p\xi} = \epsilon\,\bar{\beta}\,\bar{\rho}_p^2\,\phi_{p\xi} - \epsilon\,\bar{\alpha}\,\bar{\rho}_p^3\,\phi_\xi \tag{4.26}$$

Here

$R_e = \dfrac{\psi_o}{a_o \nu}$ , Reynolds number of the flow

$\bar{F} = \dfrac{F a_o^2}{\nu}$ , $\alpha = \dfrac{\bar{F}}{\bar{\rho}}$ , $\beta = \dfrac{\bar{F}\psi_{p_o}}{\psi_o \rho}$

$\bar{\alpha} = \dfrac{F a_o^3}{\psi_{p_o}}\,\dfrac{\psi_o}{\psi_{p_o}}\,\rho_{p_o}^2$, $\bar{\beta} = \dfrac{F a_o^3}{\psi_{p_o}}\,\rho_{p_o}$

Boundary conditions (4.19) to (4.21) become

At $\eta = S$,

$$\phi = 1,\ \phi_\eta = 0,\ \phi_p = 1 \tag{4.27}$$

At $\eta = 0$,

$$\phi = 0,\ \frac{1}{\eta}\,\phi_\xi = 0,\ \left(\frac{1}{\eta}\,\phi_\eta\right)_\eta = 0 \tag{4.28a,b,c}$$

$$\phi_p = 0,\ \frac{1}{\eta}\,\phi_{p\xi} = 0,\ \left(\frac{1}{\eta}\,\frac{1}{\bar{\rho}_p}\,\phi_{p\eta}\right)_\eta = 0 \tag{4.29a,b,c}$$

## 4.3 Series expansion

In the present analysis, we take $R_e$ to be small so that $\epsilon R_e = O(1)$. This gives low Reynolds number solution in which viscous forces dominate the non-linear inertial forces.

Here, we develop series solution in terms of the ascending powers of $\epsilon$ and calculate equations governing the first three terms. Since $\epsilon \to 0$ gives the tube of constant radius, we expand functions $\phi, \phi_p, \bar{\rho}_p$ and $q$ in asymptotic power series in $\epsilon$ as follows:

$$\phi = \phi^{(0)} + \epsilon\, \phi^{(1)} + \epsilon^2\, \phi^{(2)} + \ldots. \tag{4.30}$$

$$\phi_p = \phi_p^{(0)} + \epsilon \phi_p^{(1)} + \epsilon^2\, \phi_p^{(2)} + \ldots. \tag{4.31}$$

$$\rho_p = \bar{\rho}_p^{(0)} + \epsilon\, \bar{\rho}_p^{(1)} + \epsilon^2\, \bar{\rho}_p^{(2)} + \ldots. \tag{4.32}$$

$$q = \frac{\xi}{\epsilon} F(\xi) + P_1(\xi, \eta, \epsilon) \tag{4.33}$$

where

$$P_1(\xi, \eta, \epsilon) = q^{(0)} + \epsilon q^{(1)} + \epsilon^2 q^{(2)} + \ldots. \tag{4.34}$$

Since the zeroth order term gives the classical Poiseuille flow and for such a flow, the pressure must be a linear function of the axial coordinate. Also, in dimensional variable first term in (4.33) becomes $XF(\epsilon X)$ which has a constant gradient with respect to the variable X. Hence non-dimensional pressure $q$ can be expanded in the form of (4.33).

Substituting eqs. (4.30) to (4.34) in eqs. (4.23) to (4.26) and collecting like powers of $\epsilon$, we get the following system of differential equations governing various order terms.

$$\bar{\alpha}\ \bar{\rho}_p^{(0)^3}\phi_\xi^{(1)} - \bar{\beta}\ \bar{\rho}_p^{(0)^2}\phi_{p\xi}^{(1)} + \bar{\rho}_p^{(0)}(3\bar{\alpha}\ \phi_\xi^{(0)}\ \bar{\rho}_p^{(0)} - 2\bar{\beta}\ \phi_{p\xi}^{(0)})\ \bar{\rho}_p^{(1)}$$

$$= -\frac{1}{\eta^2}\ \bar{\rho}_p^{(0)}\phi_{p\xi}^{(0)}(\eta\,\phi_{p\xi\eta}^{(0)} - \phi_{p\xi}^{(0)}) - \frac{1}{\eta}\,\phi_{p\xi}^{(0)}(\phi_{p\eta}^{(0)}\ \bar{\rho}_{p\xi}^{(0)} - \phi_{p\xi}^{(0)}\ \bar{\rho}_{p\eta}^{(0)})$$

$$+ \frac{1}{\eta}\ \bar{\rho}_p^{(0)}\ \phi_{p\xi\xi}^{(0)}\ \phi_{p\eta}^{(0)} \qquad (4.42)$$

<u>Second order equations</u>

$$\frac{1}{\eta}\ \phi_{\eta\eta\eta}^{(2)} - \frac{1}{\eta^2}\ \phi_{\eta\eta}^{(2)} + \frac{1}{\eta^3}\ (1 - \eta^2\alpha\ \bar{\rho}_p^{(0)})\ \phi_\eta^{(2)} + \frac{\beta}{\eta}\ \phi_{p\eta}^{(2)}$$

$$- \frac{\alpha}{\eta}\ \phi_\eta^{(0)}\ \bar{\rho}_p^{(2)} = q_\xi^{(1)} + \frac{\alpha}{\eta}\ \phi_\eta^{(1)}\ \bar{\rho}_p^{(1)}$$

$$+ R_e\Big[\frac{1}{\eta^2}\ (\phi_\eta^{(0)}\ \phi_{\eta\xi}^{(1)} + \phi_\eta^{(1)}\ \phi_{\eta\xi}^{(0)}) - \frac{1}{\eta^2}\ (\phi_\xi^{(0)}\ \phi_{\eta\eta}^{(1)} + \phi_\xi^{(1)}\phi_{\eta\eta}^{(0)})$$

$$+ \frac{1}{\eta^3}\ (\phi_\xi^{(0)}\ \phi_\eta^{(1)} + \phi_\xi^{(1)}\ \phi_\eta^{(0)})\Big] - \frac{1}{\eta}\ \phi_{\eta\xi\xi}^{(0)} \qquad (4.43)$$

$$q_\eta^{(2)} = -\frac{1}{\eta^2}\ (\eta\,\phi_{\xi\eta\eta}^{(1)} - \phi_{\xi\eta}^{(1)} - \eta\,\alpha\ \bar{\rho}_p^{(0)}\ \phi_\xi^{(1)})$$

$$- \frac{1}{\eta}\ (\beta\ \phi_{p\xi}^{(1)} - \alpha\ \phi_\xi^{(0)}\ \bar{\rho}_p^{(1)})$$

$$+ \frac{R_e}{\eta^3}\ (\eta\ \phi_\eta^{(0)}\ \phi_{\xi\xi}^{(0)} - \eta\,\phi_{\xi\eta}^{(0)}\ \phi_\xi^{(0)} + \phi_\xi^{(0)^2}) \qquad (4.44)$$

$$\bar{\alpha}\ \bar{\rho}_p^{(0)^3}\phi_\eta^{(2)} - \bar{\beta}\ \bar{\rho}_p^{(0)^2}\phi_{p\eta}^{(2)} + \bar{\rho}_p^{(0)}(3\bar{\alpha}\ \bar{\rho}_p^{(0)}\,\phi_\eta^{(0)} - 2\bar{\beta}\,\phi_{p\eta}^{(0)})\ \bar{\rho}_p^{(2)}$$

$$= -3\bar{\alpha}\ \bar{\rho}_p^{(0)}\ \bar{\rho}_p^{(1)}(\phi_\eta^{(1)}\ \bar{\rho}_p^{(0)} + \phi_\eta^{(0)}\ \bar{\rho}_p^{(1)})$$

$$+ \bar{\beta}\ \bar{\rho}_p^{(1)}(\phi_{p\eta}^{(0)}\ \bar{\rho}_p^{(1)} + 2\phi_{p\eta}^{(1)}\ \bar{\rho}_p^{(0)})$$

$$+ \frac{1}{\eta}(\phi_{p\eta}^{(1)}\ \phi_{p\eta\xi}^{(0)}\ \bar{\rho}_p^{(0)} + \phi_{p\eta}^{(0)}\ \phi_{p\eta\xi}^{(1)}\ \bar{\rho}_p^{(0)} + \phi_{p\eta}^{(0)}\ \phi_{p\eta\xi}^{(0)}\ \bar{\rho}_p^{(1)})$$

$$- \frac{1}{\eta} \left( \phi_{p\eta}^{(1)} \phi_{p\eta}^{(0)} \bar{\rho}_p^{(0)} + \phi_{p\eta}^{(0)} \phi_{p\eta}^{(1)} \bar{\rho}_p^{(0)} + \phi_{p\eta}^{(0)} \phi_{p\eta}^{(0)} \bar{\rho}_{p\xi}^{(1)} \right)$$

$$+ \frac{1}{\eta} \left( \phi_{p\xi}^{(1)} \phi_{p\eta}^{(0)} \bar{\rho}_{p\eta}^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\eta}^{(1)} \bar{\rho}_{p\eta}^{(0)} + \phi_{p\xi}^{(0)} \phi_{\eta}^{(0)} \bar{\rho}_{p\eta}^{(1)} \right)$$

$$- \frac{1}{\eta} \left( \phi_{p\xi}^{(0)} \phi_{p\eta\eta}^{(0)} \bar{\rho}_p^{(1)} + \phi_{p\xi}^{(1)} \phi_{p\eta\eta}^{(0)} \bar{\rho}_p^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\eta\eta}^{(1)} \bar{\rho}_p^{(0)} \right)$$

$$+ \frac{1}{\eta^2} \left( \phi_{p\xi}^{(0)} \phi_{p\eta}^{(0)} \bar{\rho}_p^{(1)} + \phi_{p\xi}^{(1)} \phi_{p\eta}^{(0)} \bar{\rho}_p^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\eta}^{(1)} \bar{\rho}_p^{(0)} \right) \qquad (4.45)$$

$$\bar{\alpha}\, \bar{\rho}_p^{(0)^3} \phi_\xi^{(2)} - \bar{\beta}\, \bar{\rho}_p^{(0)^2} \phi_{p\xi}^{(2)} + \bar{\rho}_p^{(0)} \left( 3\bar{\alpha}\, \bar{\rho}^{(0)} \phi_\xi^{(0)} - 2\bar{\beta}\, \phi_{p\xi}^{(0)} \right) \bar{\rho}_p^{(2)}$$

$$= -3\bar{\alpha}\, \bar{\rho}_p^{(0)} \bar{\rho}_p^{(1)} \left( \phi_\xi^{(1)} \bar{\rho}_p^{(0)} + \phi_\xi^{(0)} \bar{\rho}_p^{(1)} \right) + \bar{\beta}\, \bar{\rho}_p^{(1)} \left( \phi_{p\xi}^{(0)} \bar{\rho}_p^{(1)} + 2\phi_{p\xi}^{(1)} \bar{\rho}_p^{(0)} \right)$$

$$+ \frac{1}{\eta} \left( \phi_{p\eta}^{(1)} \phi_{p\xi\xi}^{(0)} \bar{\rho}_p^{(0)} + \phi_{p\eta}^{(0)} \phi_{p\xi\xi}^{(1)} \bar{\rho}_p^{(0)} + \phi_{p\eta}^{(0)} \phi_{p\xi\xi}^{(0)} \bar{\rho}_p^{(1)} \right)$$

$$- \frac{1}{\eta} \left( \phi_{p\eta}^{(1)} \phi_{p\xi}^{(0)} \bar{\rho}_{p\xi}^{(0)} + \phi_{p\eta}^{(0)} \phi_{p\xi}^{(1)} \bar{\rho}_{p\xi}^{(0)} + \phi_{p\eta}^{(0)} \phi_{p\xi}^{(0)} \bar{\rho}_{p\xi}^{(1)} \right)$$

$$+ \frac{1}{\eta} \left( \phi_{p\xi}^{(1)} \phi_{p\xi}^{(0)} \bar{\rho}_{p\eta}^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\xi}^{(1)} \bar{\rho}_{p\eta}^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\xi}^{(0)} \bar{\rho}_{p\eta}^{(1)} \right)$$

$$+ \frac{1}{\eta} \left( \phi_{p\xi}^{(1)} \phi_{p\xi\eta}^{(0)} \bar{\rho}_p^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\xi\eta}^{(1)} \bar{\rho}_p^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\xi\eta}^{(0)} \bar{\rho}_p^{(1)} \right)$$

$$+ \frac{1}{\eta^2} \left( \phi_{p\xi}^{(1)} \phi_{p\xi}^{(0)} \bar{\rho}_p^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\xi}^{(1)} \bar{\rho}_p^{(0)} + \phi_{p\xi}^{(0)} \phi_{p\xi}^{(0)} \bar{\rho}_p^{(1)} \right) \qquad (4.46)$$

Boundary conditions (4.27) to (4.29) become

At $\eta = S$,

$$\left.\begin{array}{ll} \phi_\eta^{(n)} = 0 & \text{for } n = 0,1,2, \ldots \\ \phi^{(0)} = 1, \quad \phi^{(n)} = 0 & \text{for } n = 1,2, \ldots \\ \phi_p^{(0)} = 1, \quad \phi_p^{(n)} = 0 & \text{for } n = 1,2, \ldots \end{array}\right] \qquad (4.47a,b,c)$$

At $\eta = 0$,

$$\left.\begin{aligned} &\phi^{(n)} = 0,\ \frac{1}{\eta}\,\phi_{\xi}^{(n)} = 0,\ \left(\frac{1}{\eta}\,\phi_{\eta}^{(n)}\right)_{\eta} = 0 \\ &\qquad\qquad\qquad\qquad \text{for } n = 0,1,2,\ldots \\ &\phi_{p}^{(n)} = 0,\ \frac{1}{\eta}\,\phi_{p\xi}^{(n)} = 0 \quad \text{for } n = 0,1,2,\ldots \end{aligned}\right] \tag{4.48}$$

Boundary condition for various order terms in symmetry condition (4.29c) at the axis of the tube become

$$\left(\frac{1}{\eta}\,\frac{1}{\bar{\rho}_p^{(0)}}\,\phi_{p\eta}^{(0)}\right)_{\eta} = 0 \tag{4.49}$$

$$\left[\frac{1}{\eta}\,\frac{1}{\bar{\rho}_p^{(0)2}}\left(\bar{\rho}_p^{(0)}\,\phi_{p\eta}^{(1)} - \bar{\rho}_p^{(1)}\,\phi_{p\eta}^{(0)}\right)\right]_{\eta} = 0 \tag{4.50}$$

$$\left\{\frac{1}{\eta}\,\frac{1}{\bar{\rho}_p^{(0)3}}\left[\bar{\rho}_p^{(0)^2}\phi_{p\eta}^{(2)} - \bar{\rho}_p^{(0)}\left(\bar{\rho}_p^{(2)}\phi_{p\eta}^{(0)} + \bar{\rho}_p^{(1)}\phi_{p\eta}^{(1)}\right) + \bar{\rho}_p^{(1)^2}\phi_{p\eta}^{(0)}\right]\right\}_{\eta} = 0 \tag{4.51}$$

4.4 <u>Solution for the various order terms</u>

(i) <u>Solution of zeroth order equations</u>

Using eq. (4.37) in (4.35) and then differentiating with respect to $\eta$, we get

$$\eta^3\,\phi_{\eta\eta\eta\eta}^{(0)} - 2\eta^2\,\phi_{\eta\eta\eta}^{(0)} + 3\eta\,\phi_{\eta\eta}^{(0)} - 3\phi_{\eta}^{(0)} = 0 \tag{4.52}$$

Solution of eq. (4.52), satisfying the boundary condition (4.47a,b) and (4.48a,b,c) is

$$\phi^{(0)} = \frac{2\eta^2}{S^2} - \frac{\eta^4}{S^4} \tag{4.53}$$

In case, we solve eqs. (4.37) and (4.38), we find that zeroth order density becomes unbounded at the wall. Hence, we assume that $\bar{\rho}_p^{(0)}$= constant. Karnis, Goldsmith and Mason (1966) found experimentally that in a tube of constant cross-section, the density distribution remained uniform. Our zeroth order solution represents the flow in a tube of constant cross-section. So, the assumption $\bar{\rho}_p^{(0)}$ = constant agrees with the findings of Karnis, Goldsmith and Mason (1966). In this case, eqs. (4.37) and (4.38) become linearly dependent and the solution of $\phi_p^{(0)}$ is

$$\phi_p^{(0)} = \frac{2\eta^2}{S^2} - \frac{\eta^4}{S^4} \tag{4.54}$$

and

$$\bar{\rho}_p^{(0)} = \bar{\beta}/\bar{\alpha} = \text{const.} \tag{4.55}$$

Eqs. (4.53) and (4.54) show that the zeroth order stream functions for gas and particulate phases are identical and they represent the Poiseuille flow when S is a constant. This is in agreement with the experimental results obtained by Karnis, Goldsmith and Mason (1966).

Eq. (4.36) shows that $q^{(0)}$ is a function of $\xi$ only.

With eqs. (4.33) to (4.55), eq. (4.35) gives

$$(\xi F(\xi))_\xi = -\frac{16}{S^4} \tag{4.56}$$

(ii) Solution of the first order equations

With eqs. (4.41) and (4.53) to (4.55), eq. (4.39) becomes

$$\frac{1}{\eta}\phi^{(1)}_{\eta\eta\eta} - \frac{1}{\eta^2}\phi^{(1)}_{\eta\eta} + \frac{1}{\eta^3}\phi^{(1)}_{\eta} = q^{(0)}_{\xi} + 32R_e\left(1+\frac{\bar{\rho}^{(0)}_p}{\bar{\rho}}\right)S'\left(\frac{2\eta^2}{S^7} - \frac{\eta^4}{S^9} - \frac{1}{S^5}\right) \tag{4.57}$$

Differentiating eq. (4.57) with respect to $\eta$ and using eq. (4.36), we get

$$\eta^3\phi^{(1)}_{\eta\eta\eta\eta} - 2\eta^2\phi^{(1)}_{\eta\eta\eta} + \eta\phi^{(1)}_{\eta\eta} - 3\phi^{(1)}_{\eta} = 32R_e\left(1+\frac{\bar{\rho}^{(0)}_p}{\bar{\rho}}\right)S'\left(\frac{4\eta^5}{S^7} - \frac{4\eta^7}{S^9}\right) \tag{4.58}$$

Solution of eq. (4.58) with the boundary conditions (4.47a,b) and (4.48a,b,c) is

$$\phi^{(1)} = R_e\frac{S'}{S}\left(1+\frac{\bar{\rho}^{(0)}_p}{\bar{\rho}}\right)\left(\frac{4}{9}\frac{\eta^2}{S^2} - \frac{\eta^4}{S^4} + \frac{2}{3}\frac{\eta^6}{S^6} - \frac{1}{9}\frac{\eta^8}{S^8}\right) \tag{4.59}$$

Eliminating $\bar{\rho}^{(1)}_p$ from eqs. (4.41) and (4.42) and substituting eqs. (4.53) to (4.55) in the inhomogeneous term of the resulting equation, we get

$$\phi^{(1)}_{p\xi} + \frac{\eta S'}{S}\phi^{(1)}_{p\eta} = 16\frac{S''}{S^3}\frac{\alpha}{\bar{\beta}^2}\frac{\eta^3}{S^3}\left(1+\frac{\eta^2}{S^2}+\frac{\eta^4}{S^4}\right) - R_e\left(1+\frac{\bar{\rho}^{(0)}_p}{\bar{\rho}}\right)\left(\frac{S'^2}{S^2} - \frac{S''}{S}\right)\left(\frac{4}{9}\frac{\eta^2}{S^2} - \frac{\eta^4}{S^4} + \frac{2}{3}\frac{\eta^6}{S^6} - \frac{1}{9}\frac{\eta^8}{S^8}\right) \tag{4.60}$$

Let $S$ be a linear function of $\xi$ so that $S'' \equiv 0$

Eq. (4.60) simplifies to

$$\phi^{(1)}_{p\xi} + \frac{\eta S'}{S}\phi^{(1)}_{p\eta} = -R_e\left(1+\frac{\bar\rho^{(0)}_p}{\bar\rho}\right)\frac{S'^2}{S^2}\left(\frac{4}{9}\frac{\eta^2}{S^2} - \frac{\eta^4}{S^4} + \frac{2}{3}\frac{\eta^6}{S^6} - \frac{1}{9}\frac{\eta^8}{S^8}\right) \tag{4.61}$$

Using Monge's method, solution of the eq. (4.61) is

$$\phi^{(1)}_p = R_e\frac{S'}{S}\left(1+\frac{\bar\rho^{(0)}_p}{\bar\rho}\right)\left(\frac{4}{9}\frac{\eta^2}{S^2} - \frac{\eta^4}{S^4} + \frac{2}{3}\frac{\eta^6}{S^6} - \frac{1}{9}\frac{\eta^8}{S^8}\right) + f\left(\frac{\eta}{S}\right) \tag{4.62}$$

where f is an arbitrary function to be determined from the boundary conditions

$$f(0) = f(1) = 0,\ \dot f(0) = 0 \tag{4.63}$$

With (4.47c), (4.48d,e) and (4.63), eq. (4.62) gives

$$\phi^{(1)}_p = R_e\frac{S'}{S}\left(1+\frac{\bar\rho^{(0)}_p}{\bar\rho}\right)\left(\frac{4}{9}\frac{\eta^2}{S^2} - \frac{\eta^4}{S^4} + \frac{2}{3}\frac{\eta^6}{S^6} - \frac{1}{9}\frac{\eta^8}{S^8}\right) + \frac{\eta^4}{S^4} - \frac{\eta^2}{S^2} \tag{4.64}$$

Using eqs. (4.53) to (4.55), (4.59) and (4.64), eq. (4.41) gives

$$\bar\rho^{(1)}_p = -\frac{8S'}{\bar\beta S^3}\left(1-\frac{\eta^2}{S^2}\right) - \frac{\bar\beta}{2\bar\alpha}\left(1-\frac{2\eta^2}{S^2}\right)\left(1-\frac{\eta^2}{S^2}\right)^{-1} \tag{4.65}$$

The $\bar\rho^{(1)}_p$, in eq. (4.65) is infinite at $\eta = S$, indicating that the particles accumulate on the surface.

Using eqs. (4.53) to (4.55), eq. (4.40) gives

$$q_\eta^{(1)} = -32\eta \frac{S'}{S^5} \tag{4.66}$$

$q_\xi^{(0)}$ is known from eqs. (4.57) and (4.59).

(iii) Solution for second order equations

With eqs. (4.45), (4.53) to (4.55), (4.59), (4.64) and (4.65), eq. (4.43) becomes

$$\frac{1}{\eta}\phi^{(2)}_{\eta\eta\eta} - \frac{1}{\eta^2}\phi^{(2)}_{\eta\eta} + \frac{1}{\eta^3}\phi^{(2)}_{\eta} = q_\xi^{(1)} - \frac{8S'^2}{S^4}(3-10\frac{\eta^2}{S^2})$$
$$+16Q\,R_e\frac{S'}{S^5}\left[\gamma(1-\frac{3\eta^2}{S^2}+\frac{2\eta^4}{S^4}) - 40T\frac{S'}{S^3}(1-\frac{\eta^2}{S^2})^3\right]$$
$$-4R_e^2(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})^2\frac{S'^2}{S^6}(\frac{40}{9} - \frac{212}{9}\frac{\eta^2}{S^2} + \frac{114}{3}\frac{\eta^4}{S^4} - \frac{208}{9}\frac{\eta^6}{S^6}$$
$$+\frac{38}{9}\frac{\eta^8}{S^8}) \tag{4.67}$$

where

$$\gamma = \frac{\psi_{p_o}}{\psi_o\rho_{p_o}},\quad Q = \frac{\rho_{p_o}}{\rho},\quad T = \frac{\psi_{p_o}}{Fa_o^3\rho_{p_o}}$$

Differentiating eq. (4.67) with respect to $\eta$ and using eq. (4.66), we get

$$\eta^3\phi^{(2)}_{\eta\eta\eta\eta} - 2\eta^2\phi^{(2)}_{\eta\eta\eta} + 3\eta\phi^{(2)}_{\eta\eta} - 3\phi^{(2)}_{\eta}$$
$$=32QR_e\frac{S'}{S^2}\frac{\eta^5}{S^5}\left[120T\frac{S'}{S^3}(1-\frac{\eta^2}{S^2})^2 - \gamma(3-\frac{4\eta^2}{S^2}\right]$$
$$+Re^2(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})^2\frac{S'^2}{S^3}\frac{\eta^5}{S^5}(\frac{424}{9} - 152\frac{\eta^2}{S^2} + \frac{416}{3}\frac{\eta^4}{S^4} - \frac{384}{9}\frac{\eta^6}{S^6})$$
$$+320\frac{S'^2}{S}\frac{\eta^5}{S^5} \tag{4.68}$$

Solution of eq. (4.68) subject to the boundary conditions (4.47a,b) and (4.48a,b,c) is

$$\phi^{(2)} = \frac{5}{3} S'^2 \frac{\eta^2}{S^2} (1 - \frac{\eta^2}{S^2})^2$$

$$+ Q R_e \frac{S'}{S}(1+ \frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}) \frac{\eta^2}{S^2} [\frac{S'}{S^3} T(\frac{29}{3} - 24 \frac{\eta^2}{S^2}$$

$$+ 20 \frac{\eta^4}{S^4} - \frac{20}{3} \frac{\eta^6}{S^6} + \frac{\eta^8}{S^8}] - \gamma(\frac{5}{18} - \frac{2}{3} \frac{\eta^2}{S^2} + \frac{1}{2} \frac{\eta^4}{S^4} - \frac{1}{9} \frac{\eta^6}{S^6}]$$

$$+R_e^2 (1+ \frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})^2 \frac{S'^2}{S^2} \frac{\eta^2}{S^2} (\frac{818}{2700} - \frac{479}{540} \frac{\eta^2}{S^2} + \frac{53}{54} \frac{\eta^4}{S^4}$$

$$- \frac{19}{36} \frac{\eta^6}{S^6} + \frac{13}{90} \frac{\eta^8}{S^8} - \frac{19}{1350} \frac{\eta^{10}}{S^{10}} \qquad (4.69)$$

With eqs. (4.41), (4.53) to (4.55), (4.59), (4.64) and (4.65), eq. (4.45) becomes

$$\bar{\alpha}\phi_\eta^{(0)} \bar{\rho}_p^{(0)^2} \bar{\rho}_p^{(2)} = -[\bar{\alpha} \bar{\rho}_p^{(0)^2}(\phi_\eta^{(2)} \bar{\rho}_p^{(0)} + \phi_\eta^{(1)} \bar{\rho}_p^{(1)}) - \bar{\beta} \bar{\rho}_p^{(0)^2} \phi_{p\eta}^{(2)}]$$

$$+16 \frac{S'}{S^5} \frac{\bar{\beta}}{\bar{\alpha}} (1- \frac{3\eta^2}{S^2} + \frac{2\eta^4}{S^4}) - 640 \frac{S'^2}{S^5} \frac{1}{\bar{\beta}} (1- \frac{\eta^2}{S^2})^3$$

$$- 4 R_e \frac{\bar{\beta}}{\bar{\alpha}} (1+ \frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})(\frac{40}{9} - \frac{212}{9} \frac{\eta^2}{S^2} + \frac{114}{3} \frac{\eta^4}{S^4} - \frac{208}{9} \frac{\eta^6}{S^6}$$

$$+ \frac{38}{9} \frac{\eta^8}{S^8}) \frac{S'^2}{S^6} \qquad (4.70)$$

With eqs. (4.42), (4.53) to (4.55), (4.59), (4.64) and (4.65), eq. (4.46) becomes

$$\bar{\alpha}\,\phi_{\xi}^{(0)}\;\bar{\rho}_p^{(0)^2}\;\bar{\rho}_p^{(2)} = -\left[\bar{\alpha}\;\bar{\rho}_p^{(0)^2}\left(\phi_{\xi}^{(2)}\;\bar{\rho}_p^{(0)} + \phi_{\xi}^{(1)}\;\bar{\rho}^{(1)}\right) - \bar{\beta}\;\bar{\rho}_p^{(0)^2}\;\phi_{p\xi}^{(2)}\right]$$

$$+ 128\,\frac{S'^3}{\bar{\beta}S^7}\,\frac{\eta^2}{S^2}\,\left(1-\frac{\eta^2}{S^2}\right)\left(1+\frac{5\eta^2}{S^2}-\frac{8\eta^4}{S^4}\right)$$

$$- 8\,\frac{\bar{\beta}}{\bar{\alpha}}\,\frac{S'^2}{S^4}\,\frac{\eta^2}{S^2}\,\left(14-58\,\frac{\eta^2}{S^2}+52\,\frac{\eta^4}{S^4}\right)$$

$$+ 4\,\frac{\bar{\beta}}{\bar{\alpha}}\,\frac{S'^3}{S^5}\,R_e\left(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}\right)\left(16\,\frac{\eta^2}{S^2}-\frac{922}{9}\,\frac{\eta^4}{S^4}+\frac{588}{3}\,\frac{\eta^6}{S^6}-138\,\frac{\eta^8}{S^8}\right.$$

$$\left.+\frac{256}{9}\,\frac{\eta^{10}}{S^{10}}\right) \qquad (4.71)$$

Eliminating $\bar{\rho}_p^{(2)}$ from eqs. (4.70) and (4.71), we get

$$\phi_{p\xi}^{(2)} + \frac{\eta S'}{S}\;\phi_{p\eta}^{(2)} = 16\,\frac{S'^2}{S^4}\,\frac{T}{\gamma}\,\frac{\eta^2}{S^2}\left[\left(6-26\,\frac{\eta^2}{S^2}+24\,\frac{\eta^4}{S^4}\right)\right.$$

$$\left.+ 8\,\frac{T}{\gamma}\,\frac{S'}{S^3}\,\left(1-\frac{\eta^2}{S^2}\right)\left(4-15\,\frac{\eta^2}{S^2}+13\,\frac{\eta^4}{S^4}\right)\right]$$

$$+ R_e\;Q\,\frac{S'^2}{S^2}\,\frac{\eta^2}{S^2}\,\left[\gamma\left(\frac{5}{18}-\frac{2}{3}\,\frac{\eta^2}{S^2}+\frac{1}{2}\,\frac{\eta^4}{S^4}-\frac{1}{9}\,\frac{\eta^6}{S^6}\right)\right.$$

$$\left.-4T\,\frac{S'}{S^3}\,\left(\frac{29}{3}-24\,\frac{\eta^2}{S^2}+20\,\frac{\eta^4}{S^4}-\frac{20}{3}\,\frac{\eta^6}{S^6}+\frac{\eta^8}{S^8}\right)\right]$$

$$+ R_e\left(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}\right)\frac{S'^2}{S^2}\,\frac{\eta^2}{S^2}\left(1-\frac{\eta^2}{S^2}\right)\left(4-\frac{\eta^2}{S^2}\right)\left[\left(\frac{1}{18}-\frac{1}{9}\,\frac{\eta^2}{S^2}\right)\right.$$

$$\left.+\frac{8}{9}\,\frac{S'}{S^3}\,\frac{T}{\gamma}\,\left(1-\frac{\eta^2}{S^2}\right)^2\right]$$

$$-4\,\frac{S'}{S^3}\,\frac{T}{\gamma}\,\left(\frac{104}{9}-\frac{710}{9}\,\frac{\eta^2}{S^2}+\frac{474}{3}\,\frac{\eta^4}{S^4}-\frac{1034}{9}\,\frac{\eta^6}{S^6}+\frac{218}{9}\,\frac{\eta^8}{S^8}\right)\}$$

$$- R_e^2\left(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}\right)^2\,\frac{S'^3}{S^3}\,\frac{\eta^2}{S^2}\,\left(\frac{818}{2700}-\frac{479}{540}\,\frac{\eta^2}{S^2}+\frac{53}{54}\,\frac{\eta^4}{S^4}\right.$$

$$\left.-\frac{19}{36}\,\frac{\eta^6}{S^6}+\frac{13}{90}\,\frac{\eta^8}{S^8}-\frac{19}{1350}\,\frac{\eta^{10}}{S^{10}}\right) \quad (4.72)$$

Using Monge's method, solution of eq. (4.72) is

$$\phi_p^{(2)} = \phi^{(2)} - \frac{5}{3} S'^2 \frac{\eta^2}{S^2}\left(1-\frac{\eta^2}{S^2}\right)^2 + R_e\left(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}\right)\frac{S'}{S}\frac{\eta^2}{S^2}$$

$$\times\left\{\frac{T}{\gamma}\frac{S'}{S^3}\left(\frac{104}{9} - \frac{710}{9}\frac{\eta^2}{S^2} + \frac{474}{3}\frac{\eta^4}{S^4} - \frac{1034}{9}\frac{\eta^6}{S^6} + \frac{218}{9}\frac{\eta^8}{S^8}\right)\right.$$

$$\left. -\left(1-\frac{\eta^2}{S^2}\right)\left(4-\frac{\eta^2}{S^2}\right)\left[\left(\frac{1}{18} - \frac{1}{9}\frac{\eta^2}{S^2}\right) + \frac{2}{9}\frac{T}{\gamma}\frac{S'}{S^3}\left(1-\frac{\eta^2}{S^2}\right)^2\right]\right\}$$

$$- \frac{16}{3}\frac{T}{\gamma}\frac{S'}{S^3}\frac{\eta^2}{S^2}\left[\left(6-26\frac{\eta^2}{S^2} + 24\frac{\eta^4}{S^4}\right) + 4\frac{T}{\gamma}\frac{S'}{S^3}\left(1-\frac{\eta^2}{S^2}\right)\right.$$

$$\left.\left(4-15\frac{\eta^2}{S^2} + 13\frac{\eta^4}{S^4}\right)\right] + f_1(\eta/S) \quad (4.73)$$

where $f_1$ is an arbitrary function to be determined from the boundary conditions.

$$f_1(0) = \dot{f}_1(0) = 0, \quad f_1(1) = \frac{64}{3}\frac{\bar{\alpha}\, S'}{\bar{\beta}^2 S^3}$$

$$\frac{\ddot{f}_1}{\eta s} - \frac{\dot{f}_1}{\eta^2} \to 0 \quad \text{as} \quad \eta \to 0 \qquad (4.74)$$

conditions (4.74) give

$$f_1(\eta/S) = \frac{64}{3}\left(\frac{\bar{\alpha}\, S'^2}{\bar{\beta}^2\, S^3} - 1\right)\frac{\eta^4}{S^4} + \frac{\eta^2}{S^2} \qquad (4.75)$$

$\phi_p^{(2)}$ is now completely known from the eqs. (4.73) and (4.75).

Using eqs. (4.53), (4.59) and (4.69), eq. (4.70) gives

$$\bar{\rho}_p^{(2)} = \frac{1}{(1-\frac{n^2}{S^2})} \{ \frac{T}{4} \frac{S'^2}{S^2} R_e (1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})(\frac{96}{9} - \frac{2240}{9}\frac{n^2}{S^2} + \frac{7560}{9}\frac{n^4}{S^4} - \frac{7680}{9}\frac{n^6}{S^6} + \frac{2112}{9}\frac{n^8}{S^8})$$

$$+ \frac{\gamma}{36}\frac{S'}{S^2} R_e \,(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}) \frac{n^2}{S^2} (4 - \frac{n^2}{S^2})$$

$$- \frac{5}{6}\gamma \frac{S'^2}{S} (1-4\frac{n^2}{S^2} + 3\frac{n^4}{S^4})$$

$$-4T \frac{S'}{S^4} (3 - \frac{95}{3}\frac{n^2}{S^2} + 46\frac{n^4}{S^4})$$

$$- \frac{16}{\gamma}\frac{S'^2}{S^7} (\frac{38}{3} - \frac{166}{3}\frac{n^2}{S^2} + \frac{258}{3}\frac{n^4}{S^4} - \frac{134}{3}\frac{n^6}{S^6}) \}$$

$$+ \frac{64}{3} (\frac{T}{\gamma}\frac{S'^2}{S^3} - 1) \frac{n^4}{S^4} + \frac{n^2}{S^2} \qquad (4.76)$$

Using eq. (4.69), eq. (4.67) gives

$$q_\xi^{(1)} = - \frac{S'^2}{S^4} (\frac{88}{3} - 80\frac{n^2}{S^2})$$

$$+ Q\, R_e \frac{S'}{S^5} [256\, T \frac{S'}{S^3} - \gamma (\frac{16}{3} + \frac{20}{9}\frac{n^4}{S^4}) ]$$

$$+ \frac{1240}{135}\frac{S'^2}{S^6} R_e^2 \,(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})$$

Substituting eqs. (4.53) to (4.56), (4.59), (4.64), (4.65), (4.69), (4.73) and (4.75) in eqs. (4.30) to (4.32), we get solutions of $\phi$ , $\phi_p$ and $\bar{\rho}_p$ correct upto $\epsilon^2$.

Shear stress at the wall

Shear stress at the wall $R = a(X)$ is

$$T_w = [\sigma_{XR}(1-a_X^2) + (\sigma_{RR}-\sigma_{XX})a_X]/(1+a_X^2) \qquad (4.77)$$

where $\sigma_{RR}$, $\sigma_{XR}$ and $\sigma_{XX}$ are the stress components given by

$$\sigma_{RR} = -P + 2\rho\nu \frac{\partial V}{\partial R}$$

$$\sigma_{XX} = -P + 2\rho\nu \frac{\partial U}{\partial X}$$

$$\sigma_{XR} = \rho\nu \left(\frac{\partial U}{\partial R} + \frac{\partial V}{\partial X}\right)$$

Conditions in (4.27a,b) and (4.28a,b,c) give

At $R = a(X)$

$$\psi_R = 0, \quad \psi_{RR} = -\psi_{RX}/a_X, \quad \psi_{XX} = -a_X\, \psi_{XR} \qquad (4.78)$$

Using (4.78), eq. (4.77) becomes

$$T_w = \rho\nu \left(\frac{1}{R}\psi_{XX} - \frac{1}{R^2}\psi_R + \frac{1}{R}\psi_{RR}\right) \qquad (4.79)$$

Non-dimensional stress $\tau_w$ at the wall is given by

$$\tau_w = \frac{a_o^3 T_w}{\psi_o \rho\nu}$$

or

$$\tau_w = \frac{1}{\eta}\phi_{\eta\eta} - \frac{1}{\eta^2}\phi_\eta + \frac{\epsilon^2}{\eta}\phi_{\xi\xi}$$

$$\tau_w = -\frac{8}{S^3} + \frac{24}{9}\epsilon R_e\left(1 + \frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}\right)\frac{S'}{S^3}$$

$$+ \epsilon^2\left[\frac{8}{S^3}\left(\frac{5}{3}S'^2-1\right) + 4\frac{S'}{S^4}\, Q\, R_e\left(12T\frac{S'}{S^3} + \frac{1}{3}\gamma\right)\right.$$

$$\left. + \frac{134}{135}R_e^2\left(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}}\right)^2 \frac{S'^2}{S^5}\right] + O(\epsilon^3) \qquad (4.80)$$

Velocity Profiles

The axial velocity $\bar{U}$, for the fluid phase is given by

$$\bar{U} = \frac{1}{\eta}\phi_\eta$$

$$= \frac{4}{S^2}(1-\frac{\eta^2}{S^2}) + \epsilon\, R_e \frac{S'}{S^3}(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})$$

$$\times(\frac{8}{9} - 4\frac{\eta^2}{S^2} + 4\frac{\eta^4}{S^4} - \frac{8}{9}\frac{\eta^6}{S^6})$$

$$+\epsilon^2\Big[\frac{10}{3}\frac{S'^2}{S^2}(1-4\frac{\eta^2}{S^2} + \frac{\eta^4}{S^4}) + Re\ Q\frac{S'}{S^3}$$

$$\times\{T\frac{S'}{S^3}(\frac{58}{3} - 96\frac{\eta^2}{S^2} + 120\frac{\eta^4}{S^4} - 160\frac{\eta^6}{S^6} + 10\frac{\eta^8}{S^8})$$

$$-\gamma(\frac{10}{18} - \frac{8}{3}\frac{\eta^2}{S^2} + 3\frac{\eta^4}{S^4} - \frac{8}{9}\frac{\eta^6}{S^6})\}$$

$$+ R_e^2(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})^2\frac{S'^2}{S^3}(\frac{1636}{2700} - \frac{1916}{540}\frac{\eta^2}{S^2} + \frac{318}{54}\frac{\eta^4}{S^4}$$

$$-\frac{152}{36}\frac{\eta^6}{S^6} + \frac{130}{90}\frac{\eta^8}{S^8} - \frac{228}{1350}\frac{\eta^{10}}{S^{10}})\Big] + 0(\epsilon^3) \qquad (4.81)$$

The axial velocity $\bar{U}_p$, for the particle phase is given by

$$\bar{U}_p = \frac{1}{\eta}\ \frac{1}{\bar{\rho}_p}\ \phi_{p\eta}$$

$$\bar{U}_p = \frac{1}{\bar{\rho}_p}\{\bar{U} + \frac{2\epsilon}{S^2}(2\frac{\eta^2}{S^2} - 1) + \epsilon^2\Big[-\frac{2}{S^2} - \frac{10}{3}\frac{S'^2}{S^2}(1-4\frac{\eta^2}{S^2} + 30\frac{\eta^4}{S^4})$$

$$+ 4(\frac{64}{3}\frac{T}{\gamma}\frac{S'}{S^3} - 1)\frac{\eta^2}{S^2} - \frac{64}{3}\frac{T}{\gamma}\frac{S'}{S^5}\Big] +$$

$$\times \{(3-26\frac{\eta^2}{S^2}+36\frac{\eta^4}{S^4})+\frac{T}{\gamma}\frac{S'}{S^3}(8-76\frac{\eta^2}{S^2}+168\frac{\eta^4}{S^4}-104\frac{\eta^6}{S^6})\}$$

$$-\frac{1}{18}\frac{S'}{S^3}R_e(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})(8-52\frac{\eta^2}{S^2}+66\frac{\eta^4}{S^4}-16\frac{\eta^6}{S^6})$$

$$+\frac{S'^2}{S^6}\;\frac{T}{\gamma}R_e(1+\frac{\bar{\rho}_p^{(0)}}{\bar{\rho}})(\frac{192}{9}-\frac{2736}{9}\frac{\eta^2}{S^2}+\frac{8352}{9}\frac{\eta^6}{S^6}$$

$$-\frac{8160}{9}\frac{\eta^6}{S^6}+\frac{2160}{9}\frac{\eta^8}{S^8})]\}+O(\epsilon^3) \quad (4.82)$$

## 4.5 Discussion of the results

In this section, we present some basic features of the gas-particulate flow through a tube of varying cross-section. For the illustration of the results, the following form of the tube is considered.

$$S = 1 + .2^{\xi}$$

For numerical computation, the three forms of the series expansion (4.30) to (4.32) are considered with the following values of the parameters

$$R_e = 1-125$$
$$T = 1.0$$
$$\gamma = 1.0$$
$$\epsilon = 0.1$$
$$Q = .25$$

Fig. 1 shows the stream line pattern for gas and particle phases at $R_e = 1.0$. The figure indicates that the

stream lines are almost parallel to the surface. This figure further indicates that stream lines for the particle phase lie above the stream lines for the gas phase and as we move towards the wall, stream lines for both the phases approach closer and closer to each other. Moreover, the flow exhibits no eddy like motion at this Reynolds number. Fig. 2 gives curves of constant density in a meridian plane. This figure indicates that dust particles have tendency to move towards the wall, which results in excessive accumulation of the particles there. Due to the curvature of the pipe, centrifugal forces acting on the particles causes the particles to move towards the wall. In fact, due to the accumulation of the particles on the wall, particulate density becomes infinite there and the present analysis fails. Figs. 3 and 4 give the stream lines pattern for the two phases and constant density curves at $R_e = 10.0$. The flow variables follow the same trend as for the case $R_e = 1.0$. Fig. 3 indicates that there are no eddies in the flow even at this value of Reynolds number.

Fig. 5 gives the axial velocity distribution at $\xi = 1.0$, 2.5 and 5.0. The figure indicates that $\bar{U}_p > \bar{U}$ at all the sections of the tube, that is, particles move faster than the fluid. This figure further indicates that the difference in velocities for the gas and the particle phases decreases as we move downstream. As expected, velocities

of both the phases decrease in the downstream direction. It should be noted that for the particle phase, we have not improved the 'no slip' condition at the wall of the tube. Computation of the result show that the particulate velocity at the wall is almost equal to zero.

Fig. 6 gives the shear stress distribution for the gas phase and particle phase. The figure shows that shear stress decreases sharply in the initial portion of the tube and then, it decreases slowly in the downstream direction. This figure further shows that the shear stress for the gas phase is higher than the shear stress in gas-particulate system, that is, addition of dust particles decreases the shear stress at the wall. This result is in agreement with the results of Sproull (1961) and Saffman (1962). However, the predicted decrease in shear stress is less as compared to the observed decrease in shear stress by Sproull (1961) for the same amount of dust. This may be due to the reason that Sproull considered the turbulent flow, while the present analysis is valid for laminar flow only. In the present analysis also, Fig. 6 indicates that as the Reynolds number increases, the percentage of decrease in shear stress increases. Further, decrease in shear stress by the addition of particles implies that a given flow rate can be achieved for a gas with dust with a lesser pressure gradient than the pressure gradient needed for clean gas. This result is essentially

similar to the corresponding result of gas-particulate flow in a pipe of uniform cross-section.

Since the leading term in the present analysis represents the balance between the viscous and pressure forces, its solution, as expected, is the Poiseuille flow. Effect of the inertia terms appears as perturbation in the higher order terms. Thus the analysis from this consideration is strictly valid for low Reynolds number. In the development of the analysis of the present investigation, there is no restriction on the magnitude of Reynolds number. For higher Reynolds numbers, the contribution due to second order terms is larger than the contribution due to zeroth order and first order terms which results in formation of eddies near the upper wall of the inlet portion at $R_e$=125. Fig. 7 clearly indicates the formation of eddies. In Fig. 8, we have drawn the velocity profiles for gas and particle phases. Fig. 8a and 8b indicate the reverse flow near the wall for gas and particle phases respectively.

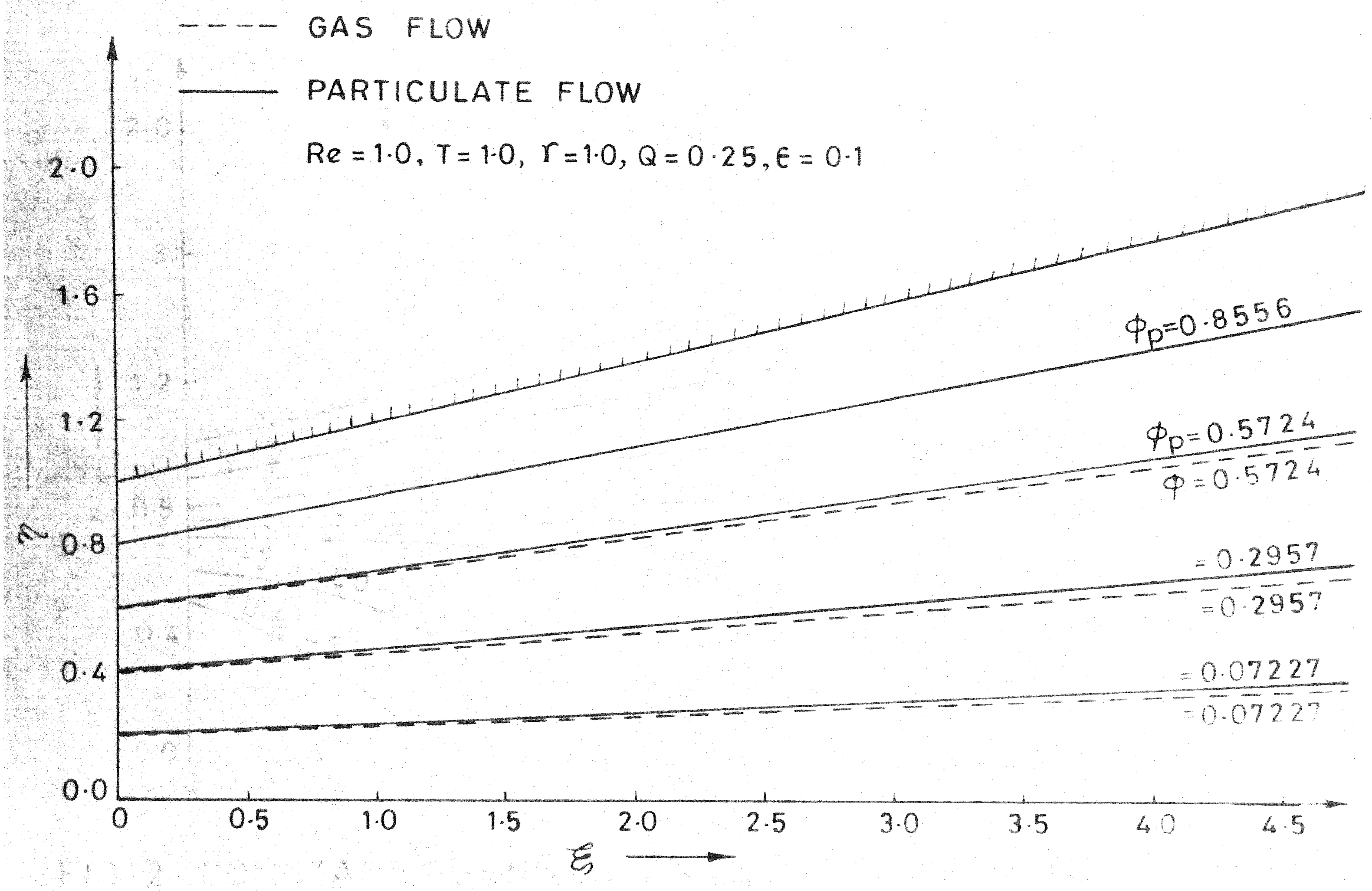



FIG. 1. STREAM LINE PATTERN FOR GAS AND PARTICLE PHASES

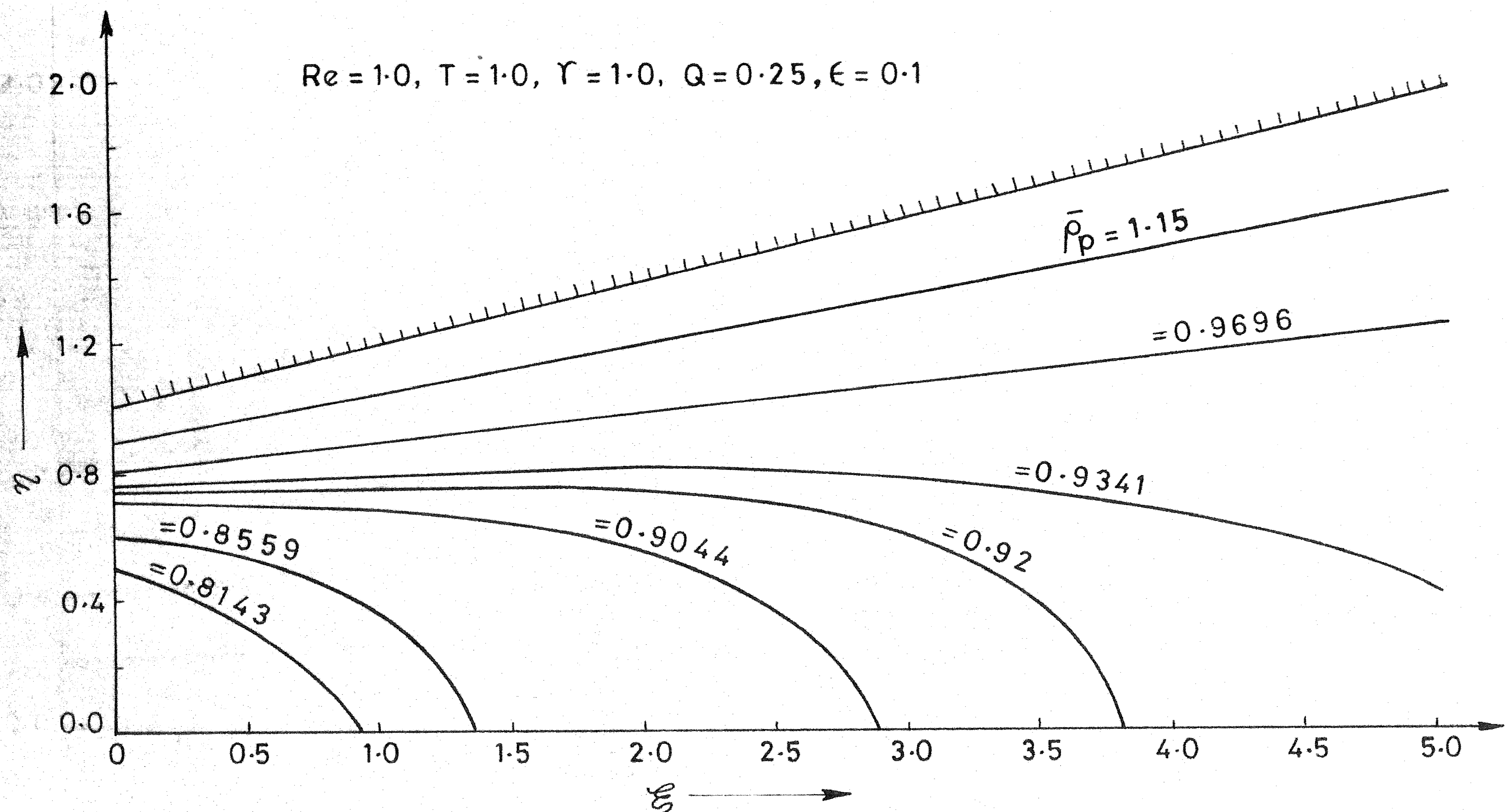



FIG. 2. CONSTANT DENSITY CURVES FOR PARTICLE PHASE.

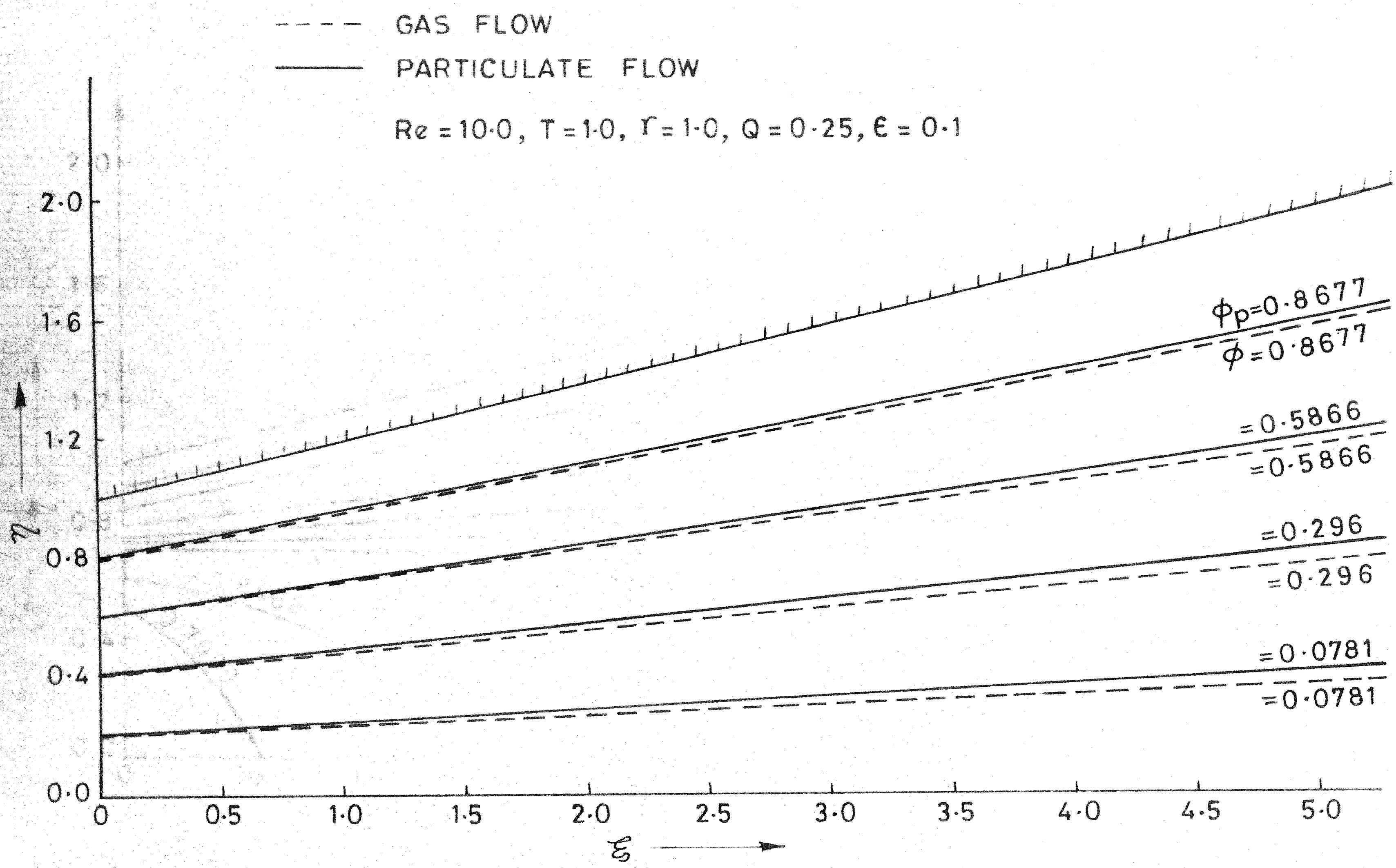



FIG. 3. STREAM LINE PATTERN FOR GAS AND PARTICLE PHASES.

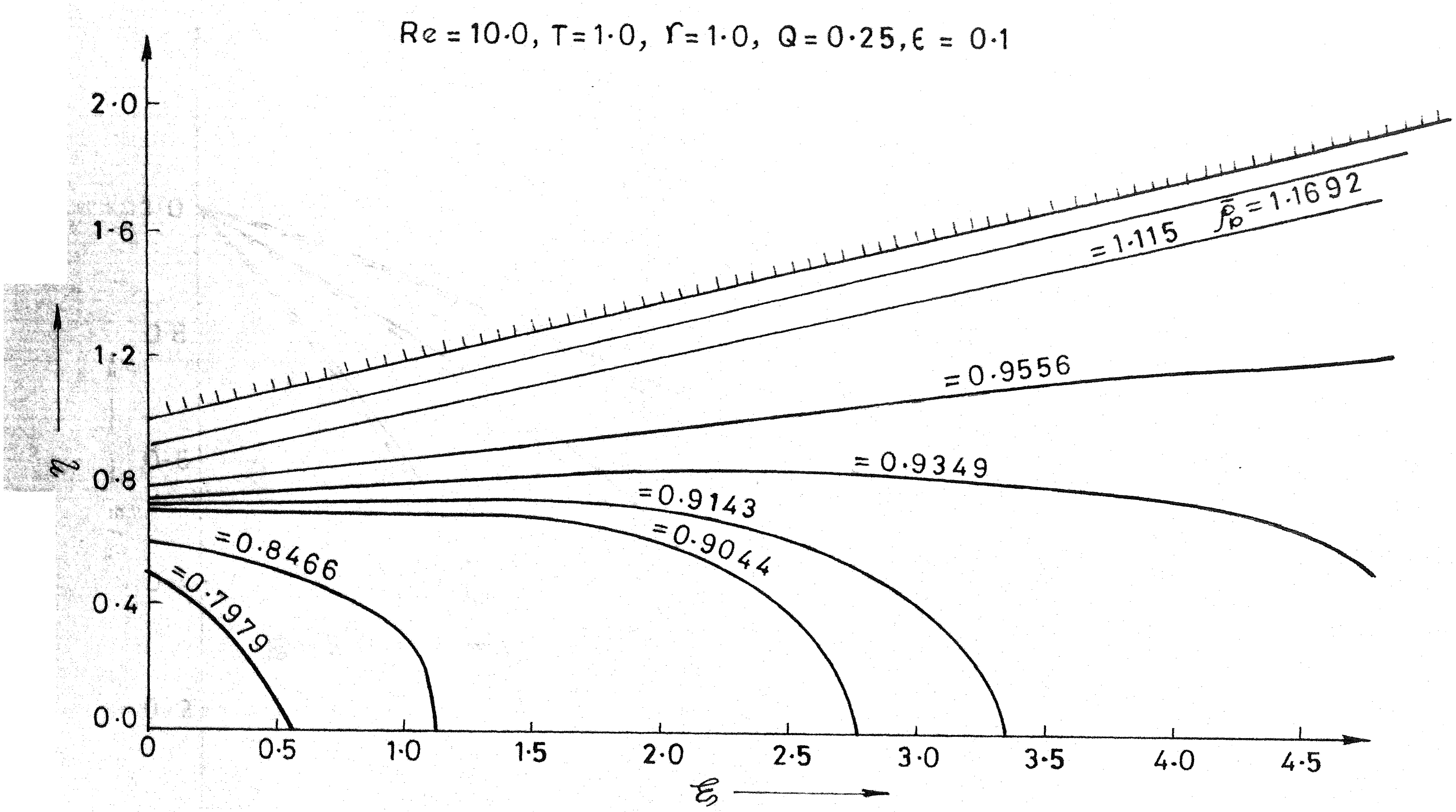



FIG. 4. CONSTANT DENSITY CURVES FOR PARTICLE PHASE.

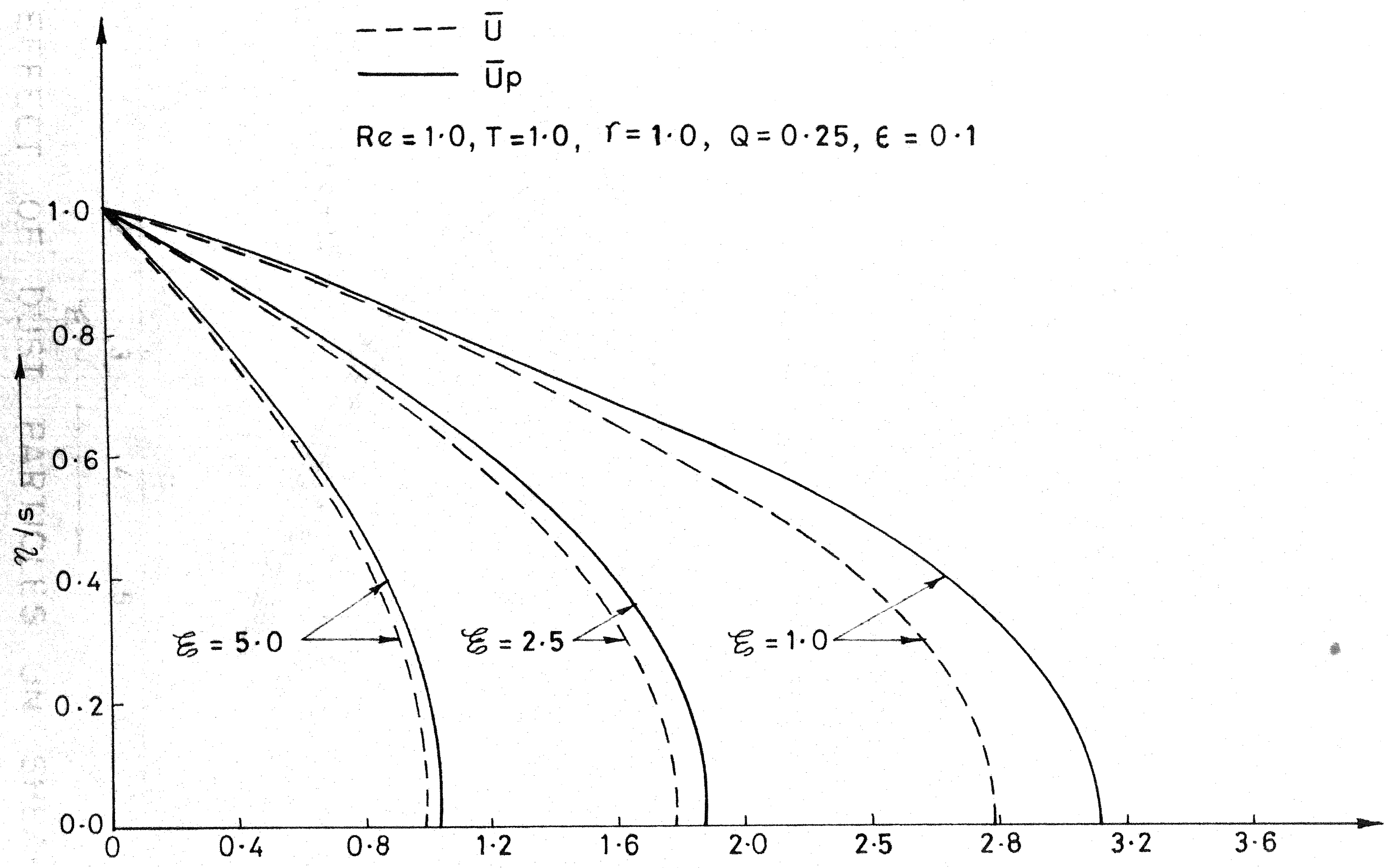
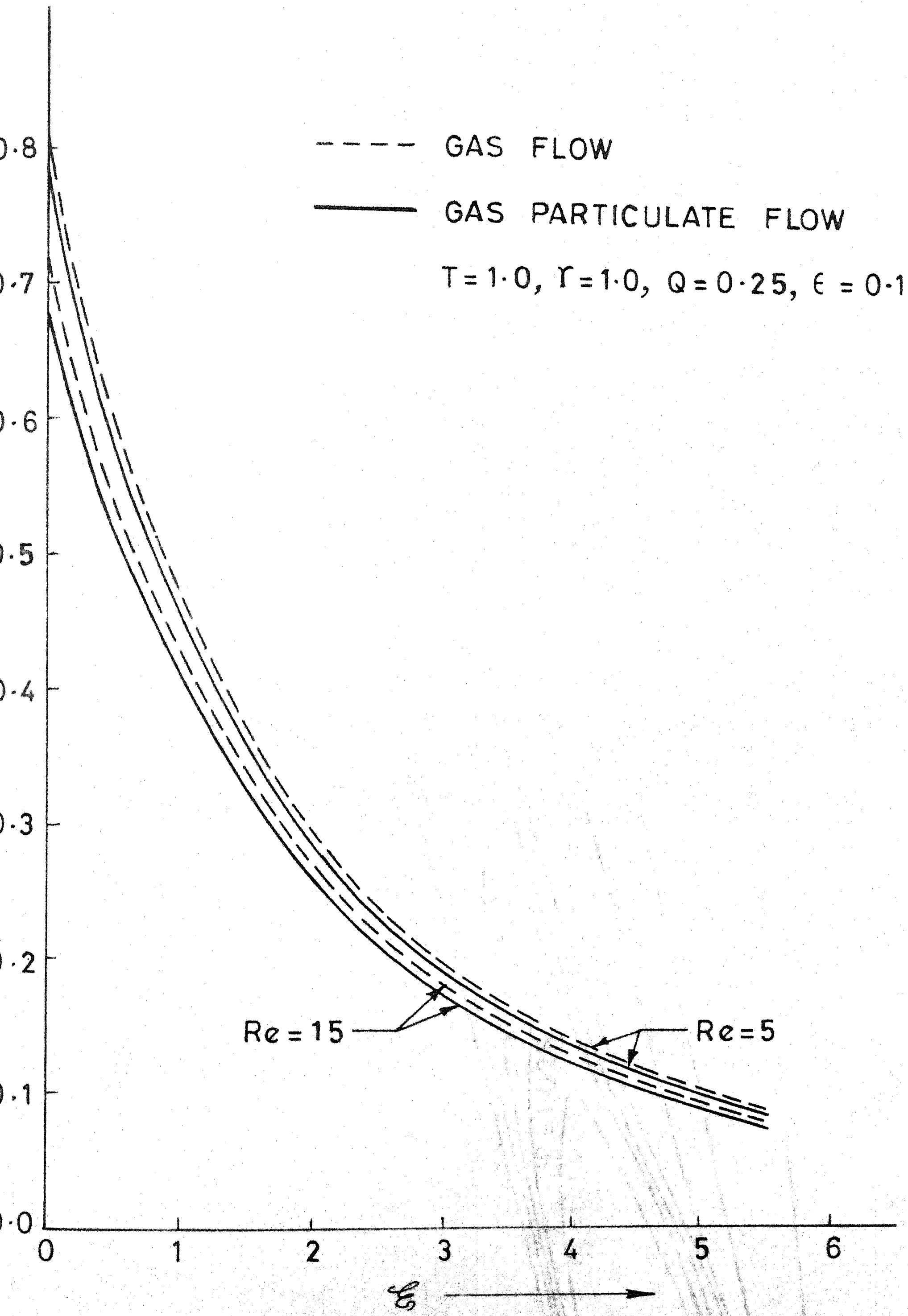



FIG. 5. AXIAL VELOCITY PROFILES FOR GAS PHASE AND PARTICLE PHASE.




. 6. EFFECT OF DUST PARTICLES ON SHEAR STRESS AT THE WALL.

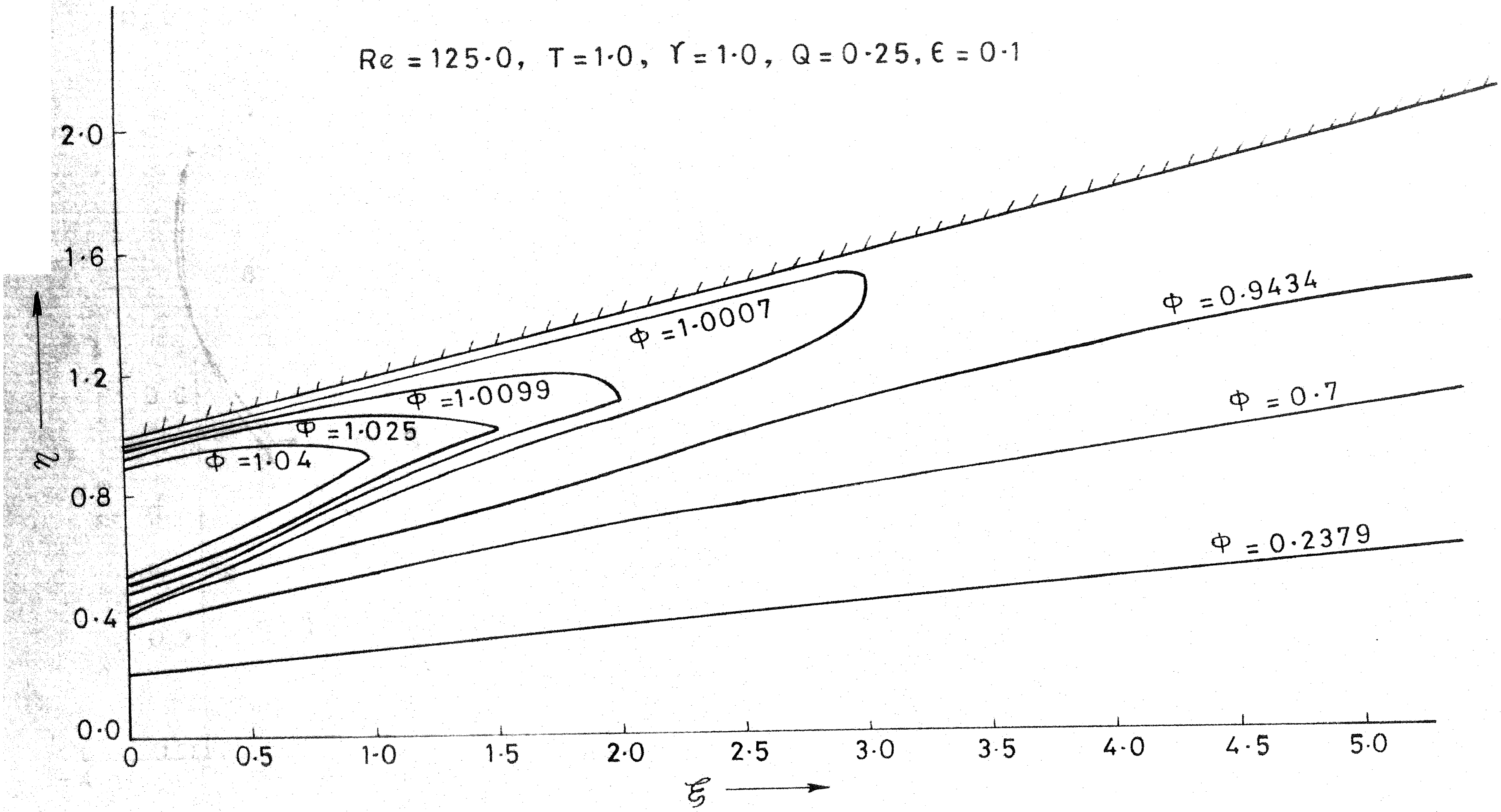



FIG. 7. STREAM LINE PATTERN FOR FLUID PHASE.

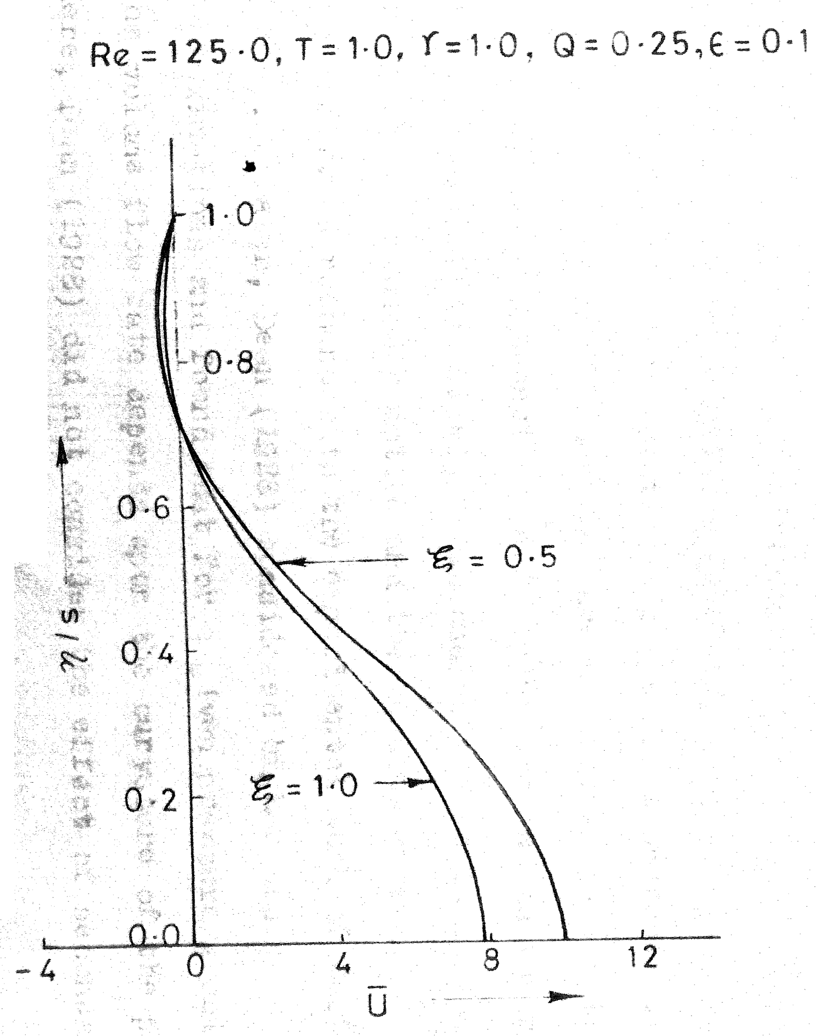



FIG. 8a. AXIAL VELOCITY DISTRIBUTION FOR GAS PHASE.

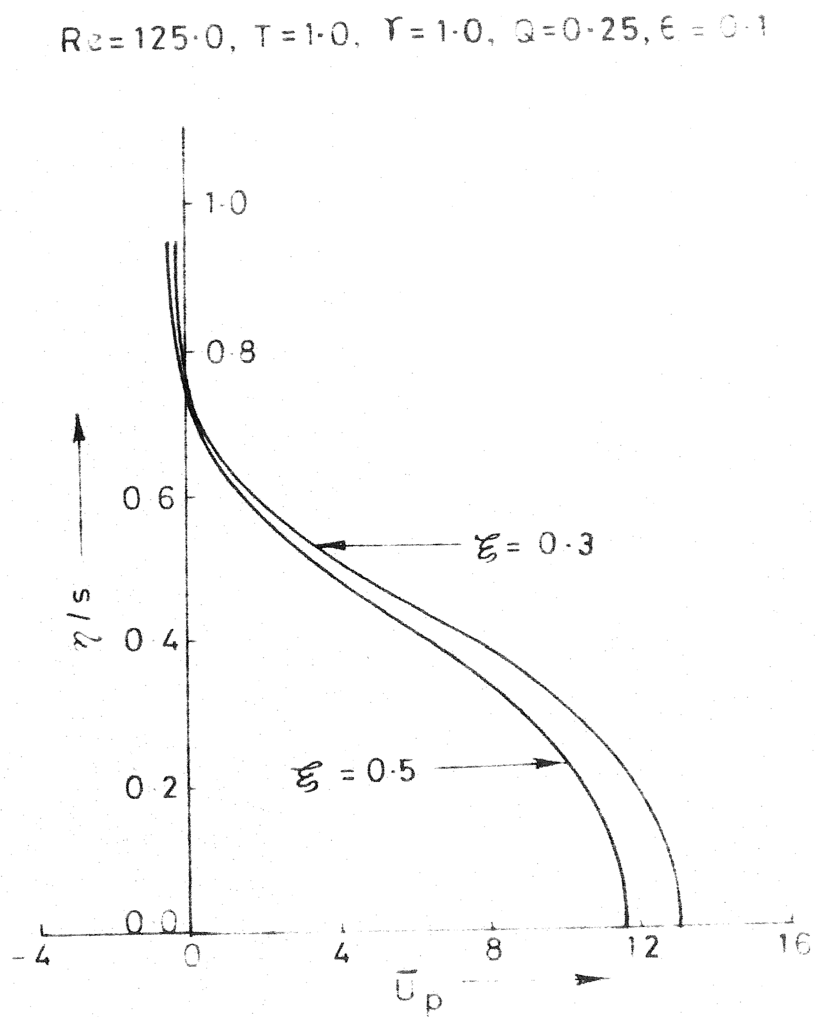



FIG. 8b. AXIAL VELOCITY DISTRIBUTION FOR PARTICLE PHASE.

## CHAPTER V

## GAS-PARTICULATE FLOW THROUGH A CURVED PIPE

### 5.1 Introduction

In a cross-section of a curved pipe, under the action of centrifugal forces and pressure gradient, a secondary motion is set up, which greatly reduces the volume flow rate. Hence to get a certain volume flow rate, a larger pressure gradient is required in a curved pipe than it is needed in the straight one. Eustice (1908) first investigated experimentally the presence of secondary motion in the flow of a curved pipe. Dean (1927) solved the Navier - Stokes equations governing the flow in a curved pipe and obtained the solution for the first approximation only (i.e. terms of order $\frac{a^2}{R^2}$ are neglected where a is the radius of the cross-section of the pipe and R is the radius of curvature of the pipe). His analysis also indicated the presence of secondary motion and the results were in qualitative agreement with the experimental work of Eustice (1908). Later, Dean (1928) considered higher order approximations and found that for a given pressure gradient, the volume flow rate depends upon the curvature of the pipe. Here, Dean (1928) did not consider the effect of secondary flow on the volume flow rate.

Dean (1959) derived analytical expression for the volume flow rate by considering the secondary flow through the curved pipe of a circular and a rectangular cross-section. Here, Dean (1959) replaced the secondary flow by a uniform stream and found that for a given pressure gradient, the secondary motion decreased the volume flow rate. He further observed that the point of maximum velocity was shifted outward. Smith (1975) obtained an asymptotic series solution of Navier-Stokes equations for large values of Dean number and for a fully developed laminar flow through the curved pipe of a circular, a rectangular and a triangular cross-section. All the above authors assumed that the radius of curvature of the pipe is large in comparison to the radius of the cross-section.

Hawthorne (1951) analyzed the flow through a bent circular pipe. He showed that secondary circulation remained unchanged if stream lines were geodesic surfaces of constant total pressure. He further observed that the secondary flow in bends was oscillatory.

To the author's knowledge, no attempt has so far been made to study the nature of the two phase flow through a curved pipe. In the present analysis, we extend the work of Dean (1959) to gas-particulate flow through a curved pipe for studying the effect of secondary motion on the volume flow rate at a given pressure gradient. We assume

that (i) the radius of the circle in which the central line of the pipe is coiled is large in comparison with the radius of the cross-section of the pipe (ii) the actual secondary motion of the gas phase is a uniform stream in the central region. We calculate the stream function for the particle phase from the governing equations. Due to the presence of the interaction term in the momentum equation for the gas phase, the method of separation of variables as adopted by Dean (1959) failed to give closed form solution. As such, we solve the governing equation by Green's function approach and obtain closed form solutions for the pipe of a circular and a rectangular cross-sections. Dean's (1959) case comes out as a particular case for the clean gas.

We find that for a given pressure gradient, the secondary flow effects the volume flow rate in the presence of the particles also. When the particles are fine in nature, the volume flow rate for the clean gas is more than the volume flow rate for the gas-particulate system. For coarse particles, the volume flow rate for the gas-particulate system is more than the volume flow rate for the gas flow. In both the cases, it is obtained that the region where the primary motion is greatest, shifts outward.

## 5.2 Mathematical formulation of the problem

Consider the steady gas-particulate flow through an infinitely long curved pipe. The radius of curvature R

of the pipe at any point is large in comparison to the radius of the cross-section a. Fig. 1a shows the system of coordinates that has been found convenient in considering the flow through a curved pipe. OZ is the axis of the anchor ring formed by the pipe wall, C is the center of the cross-section of the pipe by a plane through OZ, that makes an angle $\phi$ with a fixed plane and CO is perpendicular to OZ and is of length R. If P is any point in the cross-section and the distance of the point P from OZ is r, then the position of the point P is specified by the cylindrical coordinates $(r,\phi,z)$. The surface of the pipe is given by $r = a$, a being the radius of any section.

Here, all the flow variables except the pressure are assumed to be independent of $\phi$. If $(U,V,W)$ and $(U_p,V_p,W_p)$ are the components of velocities of the gas and particle phases respectively, then the governing equations of motion in cylindrical polar coordinates for a steady gas-particulate flow are

$$U\frac{\partial U}{\partial r} + W\frac{\partial U}{\partial z} - \frac{V^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)U - \frac{F\rho_p}{\rho}(U-U_p) \qquad (5.1)$$

$$U\frac{\partial V}{\partial r} + W\frac{\partial V}{\partial z} + \frac{UV}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \phi} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)V - \frac{F\rho_p}{\rho}(V-V_p) \qquad (5.2)$$

$$U \frac{\partial W}{\partial r} + W \frac{\partial W}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) W$$

$$- \frac{F \rho_p}{\rho} (W - W_p) \qquad (5.3)$$

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z} = 0 \qquad (5.4)$$

$$U_p \frac{\partial U_p}{\partial r} + W_p \frac{\partial U_p}{\partial z} - \frac{V_p^2}{r} = F(U - U_p) \qquad (5.5)$$

$$U_p \frac{\partial V_p}{\partial r} + W_p \frac{\partial V_p}{\partial z} + \frac{U_p V_p}{r} = F(V - V_p) \qquad (5.6)$$

$$U_p \frac{\partial W_p}{\partial r} + W_p \frac{\partial W_p}{\partial z} = F(W - W_p) \qquad (5.7)$$

$$\frac{\partial}{\partial r} (\rho_p V_p) + \frac{\rho_p U_p}{r} + \frac{\partial}{\partial z} (\rho_p W_p) = 0 \qquad (5.8)$$

Here, $\nu$, $\rho$ and P represent the kinematic viscosity, density and pressure for the gas phase respectively; $\rho_p$, $\rho_{sp}$ and d the particulate density, material density and diameter of the spherical particles respectively and F the interaction parameter given by

$$F = \frac{18\mu}{d^2 \rho_{sp}}$$

$\mu$ being the viscosity coefficient for the gas phase.

The radius of curvature R of the pipe is large as compared to the radius of the cross-section. As a consequence, it can be considered that

$$\frac{\partial}{\partial r} + \frac{1}{r} \approx \frac{\partial}{\partial r}$$

$$\frac{1}{r} \approx \frac{1}{R}$$

With these approximations, eqs. (5.1) to (5.8) reduce to the following form:

$$U \frac{\partial U}{\partial r} + W \frac{\partial U}{\partial z} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) U - \frac{F\rho_p}{\rho}(U-U_p) \quad (5.9)$$

$$U \frac{\partial V}{\partial r} + W \frac{\partial V}{\partial z} = -\frac{1}{\rho R}\frac{\partial P}{\partial \phi} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) V - \frac{F\rho_p}{\rho}(V-V_p) \quad (5.10)$$

$$U \frac{\partial W}{\partial r} + W \frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) W - \frac{F\rho_p}{\rho}(W-W_p) \quad (5.11)$$

$$\frac{\partial U}{\partial r} + \frac{\partial W}{\partial z} = 0 \quad (5.12)$$

$$U_p \frac{\partial U_p}{\partial r} + W_p \frac{\partial U_p}{\partial z} - \frac{V_p^2}{R} = F(U-U_p) \quad (5.13)$$

$$U_p \frac{\partial V_p}{\partial r} + W_p \frac{\partial V_p}{\partial z} = F(V-V_p) \quad (5.14)$$

$$U_p \frac{\partial W_p}{\partial r} + W_p \frac{\partial W_p}{\partial z} = F(W-W_p) \quad (5.15)$$

$$\frac{\partial}{\partial r}(\rho_p U_p) + \frac{\partial}{\partial z}(\rho_p W_p) = 0 \quad (5.16)$$

From eqs. (5.9) to (5.11), we get

$$P = -AR\phi + g(r,z) \quad (5.17)$$

where

$$A = -\frac{1}{R}\frac{\partial P}{\partial \phi} > 0 \qquad (5.18)$$

A gives the space rate of decrease in pressure along the central line of the pipe and comes out to be a constant because $R > a$. Due to this assumption, the factor $\frac{1}{r}$ in the term $\frac{1}{\rho r}\frac{\partial P}{\partial \phi}$ of equation (5.10) is replaced by $\frac{1}{R}$.

It is stated by Eustice (1908) that to cause a given rate of flow, a larger pressure gradient is required in a curved pipe than in a straight one, the difference being considerable even when the curvature is small. Here, there is not a constant pressure gradient as there is in the case of flow through a straight pipe. But it is natural to define the mean pressure gradient as the space rate of decrease in pressure along the central line of the pipe.

In a curved pipe, the primary motion is along the line of the pipe and the secondary motion is in the plane of the cross-section. In the present analysis, we calculate the flow rate for the secondary motion.

We define below the stream functions $\psi$ and $\psi_p$, for the secondary flow for the gas and particle phases respectively.

$$U = -\frac{\partial \psi}{\partial z}, \quad W = \frac{\partial \psi}{\partial r} \qquad (5.19a, b)$$

$$\rho_p U_p = -\frac{\partial \psi_p}{\partial z}, \quad \rho_p W_p = \frac{\partial \psi_p}{\partial r} \qquad (5.20a, b)$$

With eqs. (5.19) and (5.20), eqs. (5.12) and (5.16) are automatically satisfied.

Eq. (5.10) becomes

$$\nu\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)V = -\frac{A}{\rho} + \left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\right)V + \frac{F\rho_p}{\rho}(V-V_p) \qquad (5.21)$$

Eliminating $P$ from eqs. (5.9) and (5.11) by cross-differentiation and then using eqs. (5.19) and (5.20), we get

$$\nu\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)^2\psi = \left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2}\right) + \frac{2V}{R}\frac{\partial V}{\partial z}$$

$$+ \frac{F\rho_p}{\rho}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2}\right) - \frac{F}{\rho}\left(\frac{\partial^2\psi_p}{\partial z^2} + \frac{\partial^2\psi_p}{\partial r^2}\right)$$

$$+ \frac{F}{\rho}\left(\frac{\partial\psi}{\partial r}\frac{\partial\rho_p}{\partial r} + \frac{\partial\psi}{\partial z}\frac{\partial\rho_p}{\partial z}\right) \qquad (5.22)$$

with eqs. (5.19) and (5.20), eqs. (5.13) to (5.15) become

$$\left(\frac{\partial\psi_p}{\partial r}\frac{\partial}{\partial z} - \frac{\partial\psi_p}{\partial z}\frac{\partial}{\partial r}\right)\left(\frac{1}{\rho_p}\frac{\partial\psi_p}{\partial z}\right) + \frac{\rho_p V_p^2}{R} = F\rho_p\left(\frac{\partial\psi}{\partial z} - \frac{1}{\rho_p}\frac{\partial\psi_p}{\partial z}\right) \qquad (5.23)$$

$$\left(\frac{\partial\psi_p}{\partial r}\frac{\partial}{\partial z} - \frac{\partial\psi_p}{\partial z}\frac{\partial}{\partial r}\right)V_p = F\rho_p(V-V_p) \qquad (5.24)$$

$$\left(\frac{\partial\psi_p}{\partial r}\frac{\partial}{\partial z} - \frac{\partial\psi_p}{\partial z}\frac{\partial}{\partial r}\right)\left(\frac{1}{\rho_p}\frac{\partial\psi_p}{\partial r}\right) = F\rho_p\left(\frac{\partial\psi}{\partial r} - \frac{1}{\rho_p}\frac{\partial\psi_p}{\partial r}\right) \qquad (5.25)$$

In the approximate equations (5.21) to (5.25), the coordinates $r$ and $z$ appear only in the operators $\frac{\partial}{\partial r}$

and $\frac{\partial}{\partial z}$ , hence any point in the plane that is selected for reference can be taken as the origin, and we denote X,Y for the corresponding coordinates (Fig. 1b).

We non-dimensionalize the variables as follows:

$$\left.\begin{aligned} & X = ax,\ Y = ay,\ V = V_o v,\ V_p = V_o v_p \\ & \psi = \phi\nu\ ,\ \psi_p = \mu\phi_p,\quad \sigma_p = \frac{\rho_p}{\rho} \\ & F' = \frac{Fa^2}{\nu} \end{aligned}\right] \tag{5.26}$$

where

$$V_o = \frac{Aa^2}{\mu} \tag{5.27}$$

Here, $V_o$ is the velocity for the gas phase on the axis of the circular pipe. Velocity $V_o$ does not change due to the presence of the particles as the basic flow in a straight pipe for the gas phase as well as for the particle phase is identical [Karnis, Goldsmith and Mason (1966)].

With (5.26), eqs. (5.21) to (5.25) in the nondimensional form become

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = -4 + \left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}\right) v$$
$$+ F'\sigma_p\,(v - v_p) \tag{5.28}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi = \left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + Kv\,\frac{\partial v}{\partial y}$$

$$+F'\sigma_p\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) - F'\left(\frac{\partial^2\phi_p}{\partial x^2} + \frac{\partial^2\phi_p}{\partial y^2}\right) + F'\left(\frac{\partial\phi}{\partial x}\frac{\partial\sigma_p}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\sigma_p}{\partial y}\right) \tag{5.29}$$

$$\left(\frac{\partial \phi_p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_p}{\partial y}\frac{\partial}{\partial x}\right)\left(\frac{1}{\sigma_p}\frac{\partial \phi_p}{\partial y}\right) + \frac{K}{2} v_p^2 \sigma_p = F'\sigma_p \frac{\partial \phi}{\partial y} - F'\frac{\partial \phi_p}{\partial y} \quad (5.30)$$

$$\left(\frac{\partial \phi_p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_p}{\partial y}\frac{\partial}{\partial x}\right) v_p = F'\sigma_p(v - v_p) \quad (5.31)$$

$$\left(\frac{\partial \phi_p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_p}{\partial y}\frac{\partial}{\partial x}\right)\left(\frac{1}{\sigma_p}\frac{\partial \phi_p}{\partial x}\right) = F'\sigma_p \frac{\partial \phi}{\partial x} - F'\frac{\partial \phi_p}{\partial x} \quad (5.32)$$

where

$$K = \frac{2a}{R}\left(\frac{V_o a}{\nu}\right)^2$$

K is the measure of the curvature of the pipe and it also varies with flow Reynolds number $R_e (= \frac{V_o a}{\nu})$. Thus K is zero for a straight pipe.

In the present investigation, for the sake of simplicity, $\sigma_p$ is considered as constant. Then eqs. (5.28) to (5.32) become

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v = -4 + \left(\frac{\partial \phi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x}\right)v$$
$$+ \left(\frac{\partial \phi_p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_p}{\partial y}\frac{\partial}{\partial x}\right) v_p \quad (5.33)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi = \left(\frac{\partial \phi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + Kv\frac{\partial v}{\partial y}$$
$$+ F'\sigma_p\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) - F'\left(\frac{\partial^2 \phi_p}{\partial x^2} + \frac{\partial^2 \phi_p}{\partial y^2}\right) \quad (5.34)$$

$$\left(\frac{\partial \phi_p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_p}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial \phi_p}{\partial y} + \frac{K}{2} v_p^2 \sigma_p^2 = F'\sigma_p^2 \frac{\partial \phi}{\partial y} - F'\sigma_p \frac{\partial \phi_p}{\partial y} \quad (5.35)$$

$$\left(\frac{\partial \phi_p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \phi_p}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial \phi_p}{\partial x} = F'\sigma_p^2 \frac{\partial \phi}{\partial x} - F'\sigma_p \frac{\partial \phi_p}{\partial x} \tag{5.36}$$

Here, eq. (5.33) is obtained by eliminating $(v-v_p)$ from eqs. (5.28) and (5.31).

If the pipe is straight i.e. $K = 0$, then $\phi = \phi_p = 0$ is a solution of eq. (5.34) and eq. (5.33), then becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = -4 \tag{5.37}$$

Eq. (5.37) is the equation for the flow of liquid under pressure in a straight pipe. Eq. (5.33) for $v$ can be regarded as determining the deflection of a flexible membrane by a normal pressure, it is then clear that the pressure in the central part of the membrane will have the most important effect on the deflection. Near the center of the pipe, the secondary motion is roughly in the x-direction, the velocity $v$ being right handed about OZ. This suggests replacing the actual secondary motion by a uniform stream in the x-direction. We choose

$$\phi = -2ky \tag{5.38}$$

where $k$ is a constant. The boundary conditions at the surface of the pipe are not satisfied by the assumed secondary motion.

Eq. (5.36), on using (5.38) becomes

$$\frac{\partial \phi_p}{\partial x} \frac{\partial^2 \phi_p}{\partial x \partial y} - \frac{\partial \phi_p}{\partial y} \frac{\partial^2 \phi_p}{\partial x^2} + F' \sigma_p \frac{\partial \phi_p}{\partial x} = 0 \qquad (5.39)$$

Using Monge's method, we get a solution of the eq. (5.39) as

$$\phi_p = \left(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda x}\right) y \qquad (5.40)$$

where $C$ and $\lambda$ are arbitrary constants.

For $\phi_p$, given by eq. (5.40), two cases are of particular interest, namely:

Case I

$$\left|\frac{F'\sigma_p}{\lambda}\right| \gg Ce^{\lambda x}$$

which gives

$$\phi_p \approx \frac{F'\sigma_p}{\lambda} y$$

Thus $\phi_p$, for $\lambda$ negative, is of the same nature as $\phi$ i.e. secondary flow for particle phase can be replaced by a uniform stream. This case corresponds to the consideration of fine particles.

Case II

$$\left|\frac{F'\sigma_p}{\lambda}\right| \ll Ce^{\lambda x}$$

It gives

$$\phi_p \approx Ce^{\lambda x} y$$

which shows that stream function for the secondary flow for

The eigen functions of the homogeneous boundary value problem

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - 2k\,\frac{\partial v}{\partial x} = 0 \tag{5.43}$$

with boundary conditions

$$v = 0 \quad \text{at} \quad x = \pm 1 \quad \text{and} \quad y = \pm b \tag{5.44}$$

are

$$\cos\,(m + \tfrac{1}{2})\,\frac{\pi y}{b}\,, \quad m = 0,1,2,\ldots$$

These eigen functions are useful in the construction of Green's function.

Let $G(x,y;\xi,\eta)$ be the Green's function for the homogeneous boundary value problem defined by eqs. (5.43) and (5.44), then $G$ satisfies the equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} - 2k\,\frac{\partial G}{\partial x} = -\delta(x-\xi)\,\delta(y-\eta) \tag{5.45}$$

such that

$$G = 0 \quad \text{at} \quad x = \pm 1 \quad \text{and} \quad y = \pm b \tag{5.46}$$

For each fixed $x$, we expand $G$ in a series in eigen functions $\cos\,(m + \frac{1}{2})\,\frac{\pi y}{b}$ and the coefficients then depend on $x, \xi, \eta$. To simplify the notation, we suppress the dependence on $x, \xi, \eta$

Let

$$G = \sum_{m=0}^{\infty} G_m \cos\,(m + \tfrac{1}{2})\,\frac{\pi y}{b} \tag{5.47}$$

where

$$G_m = \frac{1}{b} \int_{-1}^{1} G \cos\,(m + \tfrac{1}{2}) \frac{\pi y}{b}\, dy \tag{5.48}$$

Since $G = 0$ at $x = \pm 1$ and $y = \pm b$, eq. (5.48) gives

$$G_m = 0 \quad \text{at} \quad x = \pm 1,\ m = 0, 1, 2, \ldots \tag{5.49}$$

Multiplying eq. (5.45) by $\frac{1}{b} \cos\,(m + \frac{1}{2}) \frac{\pi y}{b}$ and then integrating from $-b$ to $b$, we get

$$\frac{\partial^2 G_m}{\partial x^2} - (m + \tfrac{1}{2})^2 \frac{\pi^2}{b^2} G_m - 2k \frac{\partial G_m}{\partial x} = -\frac{1}{b} \cos(m + \tfrac{1}{2}) \frac{\pi \eta}{b}\, \delta(x - \xi) \tag{5.50}$$

Self adjoint form of eq. (5.50) is

$$\frac{\partial}{\partial x} \left[\frac{b e^{-2kx}}{\cos \mu_m \eta} \frac{\partial G_m}{\partial x}\right] - \frac{\mu_m^2\, b}{\cos \mu_m \eta} e^{-2kx}\, G_m = -\delta(x - \xi) e^{-2kx} \tag{5.51}$$

where

$$\mu_m = \cos\,(m + \tfrac{1}{2}) \frac{\pi}{b}$$

In order to find $G_m$ given by eq. (5.51), satisfying the boundary condition (5.49), again the method of Green's function is used. Hence the problem of constructing two dimensional Green's function has reduced to the construction of one dimensional Green's function. We construct now, the Green's function for the homogeneous boundary value problem

$$\frac{\partial}{\partial x} \left[\frac{b}{\cos \mu_m \eta} e^{-2kx} \frac{\partial G_m}{\partial x}\right] - \frac{b \mu_m^2\, e^{-2kx}}{\cos \mu_m \eta}\, G_m = 0 \tag{5.52}$$

satisfying the boundary condition in (5.49).

If $\bar{G}_m(x, \xi')$ is the required Green's function for eq. (5.52) satisfying (5.49), then

$$\frac{\partial}{\partial x}\left[\frac{b}{\cos \mu_m \eta} e^{-2kx} \frac{\partial \bar{G}_m}{\partial x}\right] - \frac{b\mu_m^2 e^{-2kx}}{\cos \mu_m \eta} \bar{G}_m = -\delta(x-\xi') \quad (5.53)$$

where

$$\bar{G}_m = 0 \quad \text{at} \quad x = \pm 1$$

The fundamental solutions of the homogeneous part of eq. (5.53) are

$$e^{kx} \sinh \sigma_m x \quad \text{and} \quad e^{kx} \cosh \sigma_m x$$

where

$$\sigma_m = \sqrt{k^2 + \mu_m^2}$$

$$\therefore \quad \bar{G}_m(x, \xi') = e^{kx}(\bar{A} \sinh \sigma_m x + \bar{B} \cosh \sigma_m x) \text{ for } -1 \leq x < \xi'$$

$$= e^{kx}(\bar{C} \sinh \sigma_m x + \bar{D} \cosh \sigma_m x) \text{ for } \xi' < x \leq 1$$

$$(5.54)$$

such that

$$\bar{G}_m(-1) = \bar{G}_m(1) = 0$$

$$\bar{G}_m(\xi'_+) = \bar{G}_m(\xi'_-)$$

$$\bar{G}'_m(\xi'_+) - \bar{G}'_m(\xi'_-) = -\frac{1}{e^{-2k\xi'}} \frac{\cos \mu_m \eta}{b}$$

where $\bar{G}'_m$ is the derivative of $\bar{G}_m$ with respect to x.

Eq. (5.54), satisfying the conditions in (5.55) gives

$$\bar{G}_m(x,\xi') = -\frac{\cos\mu_m\eta}{b\sigma_m\sinh 2\sigma_m} e^{k(x+\xi')}\sinh\sigma_m(\xi'-1)\sinh\sigma_m(x+1)$$

$$\text{for } -1 \leq x < \xi'$$

$$= -\frac{\cos\mu_m\eta}{b\sigma_m\sinh 2\sigma_m} e^{k(x+\xi')}\sinh\sigma_m(x-1)\sinh\sigma_m(\xi+1)$$

$$\text{for } \xi' < x \leq 1 \quad (5.56)$$

The solution of eq. (5.51) is

$$G_m = \int_{-1}^{1} \delta(\xi'-\xi)\, e^{-2k\xi'}\, \bar{G}_m(x,\xi')\, d\xi'$$

or $$G_m = -\frac{\cos\mu_m\eta}{b\sigma_m\sinh 2\sigma_m} e^{-k(\xi-x)} \sinh\sigma_m(\xi-1)\sinh\sigma_m(x+1)$$

$$\text{for } -1 \leq x < \xi$$

$$= -\frac{\cos\mu_m\eta}{b\sigma_m\sinh 2\sigma_m} e^{-k(\xi-x)}\sinh\sigma_m(x-1)\sinh\sigma_m(\xi+1)$$

$$\text{for } \xi < x \leq 1 \quad (5.57)$$

Hence eqs. (5.47) and (5.57) give $G$, the complete solution of the system of eqs. (5.45) and (5.46)

5.3b <u>For circular cross-section</u>

Suppose that the cross-section of the pipe is a circle of radius $a$ and let $x = r\cos\theta$ and $y = r\sin\theta$ so that $r = 1$ is the boundary of the section.

In eq. (5.42), we substitute

$$v = ue^{kx} \quad (5.58)$$

and get the resulting equation as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - k^2 u = 4e^{-kx} - \frac{C^2\lambda^2(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda x})e^{(2\lambda-k)x}}{K^{1/2}[2C^2\lambda e^{2\lambda x} - 4k\sigma_p^2 F' - 2F'^2\sigma_p^2/\lambda]^{1/2}}$$

Changing it to polar-coordinates, we get .

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - k^2 u = -4e^{-kr\cos\theta}$$

$$- \frac{C^2\lambda^2(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda r\cos\theta})\, e^{(2\lambda-k)r\cos\theta}}{K^{1/2}[2C^2\lambda e^{2\lambda r\cos\theta} - 4k\sigma_p^2 F' - 2F'^2\sigma_p^2/\lambda]^{1/2}} \qquad (5.59)$$

The eigen functions of the homogeneous boundary value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - k^2 u = 0 \qquad (5.60)$$

with

$$u = 0 \quad \text{at} \quad r = 1 \qquad (5.61)$$

are

$$\cos n\theta,\ n = 0,1,2,\ldots$$

If $G(r,\theta;r_o,\theta_o)$ is the Green's function for the boundary value problem defined by eqs. (5.60) and (5.61), then

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} - k^2 G = -\frac{1}{r}\,\delta(r-r_o)\delta(\theta-\theta_o) \qquad (5.62)$$

such that

$$G = 0 \quad \text{at} \quad r = 1 \tag{5.63}$$

Let

$$G = \sum_{n=0}^{\infty} G_n \cos n\theta \tag{5.64}$$

where $G_n'$s are the functions of $r, r_o$ and $\theta_o$ and are given by

$$\left.\begin{aligned} G_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} G \cos n\theta \, d\theta, \quad n=1,2,3,\ldots \\ \text{and} \qquad & \\ G_o &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G \cos n\theta \, d\theta, \quad n=0 \end{aligned}\right] \tag{5.65}$$

Eq. (5.65), on making use of (5.63) gives

$$G_n = 0 \quad \text{at} \quad r = 1 \quad \text{for all } n \tag{5.66}$$

For $n \neq 0$, multiplying eq. (5.62) by $\frac{1}{\pi} \cos n\theta$, and then integrating from $-\pi$ to $\pi$, we get

$$\frac{\partial}{\partial r}\left[r \frac{\partial G_n}{\partial r}\right] - \left(\frac{n^2}{r} + k^2\right) G_n = -\frac{1}{\pi} \delta(r - r_o) \cos n\theta_o \tag{5.67}$$

The fundamental solutions of the equation

$$\frac{\partial}{\partial r}\left[r \frac{\partial G_n}{\partial r}\right] - \left(\frac{n^2}{r} + k^2\right) G_n = 0$$

are the modified Bessel functions, $I_n(kr)$ and $K_n(kr)$, given by

$$I_n(kr) = \sum_{m=0}^{\infty} \frac{(kr)^{2m+n}}{2^{2m+n} \, m!(m+n)!} \tag{5.68}$$

and

$$K_n(kr) = (-1)^{n+1} I_n(kr) \log(\tfrac{kr}{2})$$

$$+ (-\tfrac{kr}{2})^n \tfrac{1}{2} \sum_{m=0}^{\infty} \frac{1}{m!(m+n)!}(\tfrac{kr}{2})^{2m} \left[\frac{\Gamma'(m+1)}{\Gamma(m+1)} + \frac{\Gamma'(m+n+1)}{\Gamma(m+n+1)}\right]$$

$$+ \tfrac{1}{2} \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} (\tfrac{kr}{2})^{2m-n} \tag{5.69}$$

$K_n(kr)$ is infinite at $r = 0$, therefore $I_n(kr)$ is taken as the solution for $r < r_o$. Hence

$$\left.\begin{aligned} G_n(r,r_o) &= \alpha_n'(I_n(kr)) && \text{for } 0 \leq r < r_o \\ &= \alpha_n I_n(kr)+\beta_n K_n(kr) && \text{for } r_o < r \leq 1 \end{aligned}\right] \tag{5.70}$$

Conditions on $G_n(r,r_o)$ are

$$\left.\begin{aligned} G_n(1) &= 0 \\ G_n(r_{o+}) &= G_n(r_{o-}) \\ G_n'(r_{o+}) &- G_n'(r_{o-}) = -\frac{1}{\pi r_o} \cos n\theta_o \end{aligned}\right] \tag{5.71}$$

where ' denotes the derivative with respect to r.

Eq. (5.70), with conditions in (5.71), and using

$$r_o[I_n(kr_o) K_n'(kr_o) - I_n'(kr_o) K_n(kr_o)] = \text{const.} = -1$$

gives, for $n \neq 0$

$$G_n(r,r_o) = \frac{\cos n\theta_o}{k\pi I_n(k)} I_n(kr)[I_n(k)K_n(kr_o)-I_n(kr_o)K_n(k)]$$
$$\text{for } 0 \leq r < r_o$$

$$= \frac{\cos n\theta_o}{k\pi I_n(k)} I_n(kr_o)[I_n(k)K_n(kr)-I_n(kr)K_n(k)]$$
$$\text{for } r_o < r \leq 1 \tag{5.72}$$

Along the same lines for $n = 0$, we get

$$G_o(r,r_o) = \frac{I_o(kr)}{2k\pi I_o(k)} [I_o(k)K_o(kr_o) - I_o(kr_o)K_o(k)]$$
$$\text{for } 0 \leq r < r_o$$

$$= \frac{I_o(kr_o)}{2k\pi I_o(k)} [I_o(k)K_o(kr) - I_o(kr)K_o(k)]$$
$$\text{for } r_o < r \leq 1 \qquad (5.73)$$

The complete solution $G$ of eq. (5.60) satisfying the boundary condition (5.61) is now known from eq. (5.64).

## 5.4 Solution of the problem

### 5.4a For rectangular cross-section

Solution of the eq. (5.42) subject to the boundary conditions

$$v = 0 \quad \text{at} \quad x = +1 \quad \text{and} \quad y = \pm b$$

is

$$v(x,y) = \int_{-1}^{1} \int_{-b}^{b} \{ [4 + \frac{2C^2\lambda^2(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda\xi})\, e^{2\lambda\xi}}{K^{1/2}[2C^2\lambda e^{2\lambda\xi} - 4k\sigma_p^2 F' - 2F'^2\sigma_p^2/\lambda]^{1/2}}]$$
$$\times G(x,y;\xi,\eta)\} d\xi d\eta \qquad (5.74)$$

The solution corresponding to the first term in the integrand in eq. (5.74) represents the case of clean gas.

Using the identities

(i) $\int e^{ax}\sinh bx dx = \frac{e^{ax}}{a^2-b^2}(a \sinh bx - b \cosh bx)$

(ii) $$y^2-b^2 = \sum_{m=0}^{\infty} \frac{4(-1)^m \cos \mu_m y}{b\mu_m^3}$$

the solution corresponding to the clean gas

$$= 2(b^2-y^2) + 8e^{kx} \sum_{m=0}^{\infty} \{ \frac{(-1)^m \sinh k \cos \mu_m y \sinh \sigma_m x}{b\mu_m^3 \sinh \sigma_m}$$

$$+ \frac{(-1)^{m+1} \cosh k \cos \mu_m y \cosh \sigma_m x}{b\mu_m^3 \cosh \sigma_m} \} \qquad (5.75)$$

This solution is the same as obtained by Dean (1959) by the method of separation of variables.

The solution corresponding to the second term in the integrand in eq. (5.74) represents the case of the particulate flow. Here, we integrate this term in the following cases:

Case I

$$\frac{F'\sigma_p}{\lambda} \gg Ce^{\lambda x}, \quad \lambda \text{ negative.}$$

Then the expression

$$\frac{2C^2\lambda^2 \left(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda\xi}\right) e^{2\lambda\xi}}{\left[2C^2 \lambda e^{2\lambda\xi} - 4k\sigma_p^2 F' - 2F'^2 \sigma_p^2/\lambda\right]^{1/2}}$$

$$= \frac{2C^2\lambda^2 \left(\frac{F'}{\lambda} + \frac{C}{\sigma_p} e^{\lambda\xi}\right) e^{2\lambda\xi}}{\left[-2F'(2k+F'/\lambda)\right]^{1/2}}, \text{ after neglecting}$$

second order terms.

With this simplification, the solution corresponding to the particulate flow, given by the second term in the integrand of eq. (5.74), is

$$= - \sum_{m=0}^{\infty} \frac{4D\,(-1)^m F' C^2 \lambda \cos \mu_m y\, e^{kx}}{b\mu_m \sinh 2\sigma_m \left[-2KF'(2k+\frac{F'}{\lambda})\right]^{1/2} \left[(2\lambda-k)^2 - \sigma_m^2\right]}$$

$$= - \sum_{m=0}^{\infty} \frac{4E\,(-1)^m C^3 \lambda^2 \cos \mu_m y\, e^{kx}}{b\sigma_p \mu_m \sinh 2\sigma_m \left[-2KF'(2k+\frac{F'}{\lambda})\right]^{1/2} \left[(3\lambda-k)^2 - \sigma_m^2\right]}$$

$$- \sum_{m=0}^{\infty} \frac{4(-1)^m C^2 \lambda F' \cos \mu_m y\, e^{2\lambda x}}{b\mu_m \left[-2KF'(2k+F'/\lambda)\right]^{1/2} \left[(2\lambda-k)^2 - \sigma_m^2\right]}$$

$$- \sum_{m=0}^{\infty} \frac{4(-1)^m C^3 \lambda^2 \cos \mu_m y\, e^{3\lambda x}}{b\sigma_p \mu_m \left[-2KF'(2k+F'/\lambda)\right]^{1/2} \left[(3\lambda-k)^2 - \sigma_m^2\right]} \tag{5.76}$$

where

$$D = e^{-(2\lambda-k)} \sinh \sigma_m(x-1) - e^{(2\lambda-k)} \sinh \sigma_m(x+1)$$

and

$$E = e^{-(3\lambda-k)} \sinh \sigma_m(x-1) - e^{(3\lambda-k)} \sinh \sigma_m(x+1)$$

The complete solution $v(x,y)$ of the eq. (5.74) is now the sum of the expressions (5.75) and (5.76).

The total rate of flow through the pipe depends on the integral of $v$ taken over the area of the cross-section.

$$\text{Volume rate of flow} = F_1(\text{say}) = \int_{-1}^{1} \int_{-b}^{b} v(x,y)\,dx\,dy$$

After doing some algebraic simplifications, we get

$$F_1 = \frac{16b^3}{3} - \frac{16}{b} \sum_{m=0}^{\infty} \frac{\sigma_m(\cosh 2\sigma_m - \cosh 2k)}{\mu_m^6 \sinh \sigma_m \cosh \sigma_m}$$

$$- \sum_{m=0}^{\infty} \frac{16F'LC^2\lambda}{b\mu_m^4 \sinh 2\sigma_m \left[-2KF'(2k+\frac{F'}{\lambda})\right]^{1/2} \left[(2\lambda-k)^2 - \sigma_m^2\right]}$$

$$- \sum_{m=0}^{\infty} \frac{16MC^3\lambda^2}{b\sigma_p\mu_m^4 \sinh 2\sigma_m \left[-2KF'(2k+\frac{F'}{\lambda})\right]^{1/2} \left[(3\lambda-k)^2 - \sigma_m^2\right]}$$

$$- \sum_{m=0}^{\infty} \frac{8C^2\lambda}{b\mu_m^2 \left[-2KF'(2k+\frac{F'}{\lambda})\right]^{1/2}} \left[ \frac{F' \sinh 2\lambda}{\lambda\left[(2\lambda-k)^2 - \sigma_m^2\right]} + \frac{2C \sinh 3\lambda}{3\sigma_p\left[(3\lambda-k)^2 - \sigma_m^2\right]} \right] \quad (5.78)$$

where

$$L = k \sinh 2\sigma_m \sinh 2\lambda - \sigma_m \cosh 2\sigma_m \cosh 2\lambda + \sigma_m \cosh(2\lambda - 2k)$$

and

$$M = k \sinh 2\sigma_m \sinh 3\lambda - \sigma_m \cosh 2\sigma_m \cosh 3\lambda + \sigma_m \cosh(3\lambda - 2k)$$

The first two terms of the eq. (5.78) give the case of particle free fluid which agrees with Dean's (1959) results.

<u>Case II</u>

$$\frac{F'\sigma_p}{\lambda} << Ce^{\lambda x}, \ C \text{ negative.}$$

Then the expression

$$\frac{2C^2\lambda^2(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda x})\, e^{2\lambda x}}{\left[2C^2 \lambda\, e^{2\lambda x} - 4k\,\sigma_p^2\, F' - 2F'^2\sigma_p^2/\lambda\right]^{1/2}}$$

$$= \sqrt{\frac{2\lambda^3}{K}}\, Ce^{\lambda x}\, (\frac{F'\sigma_p}{\lambda} + Ce^{\lambda x}) + \sqrt{\frac{\lambda^3}{2K}}\, C\, (\frac{F'\sigma_p^2}{C^2\lambda^2} + \frac{2kF'\sigma_p^2}{C^2\lambda})$$

$$(\frac{F'\sigma_p}{\lambda}\, e^{-\lambda x} + C)$$

With this simplification, the solution corresponding to the particulate flow, given by the second term in the integrand of eq. (5.74), is

$$= -\sum_{m=0}^{\infty} \frac{2(-1)^m\sqrt{2\lambda^3/K}\; Ce^{kx}\, \cos\mu_m y\, F'\sigma_p}{b\lambda\mu_m\left[(\lambda-k)^2 - \sigma_m^2\right]\sinh 2\sigma_m}$$

$$\times \left[e^{-(\lambda-k)}\sinh\sigma_m(x-1) - e^{(\lambda-k)}\sinh\sigma_m(x+1)\right]$$

$$-\sum_{m=0}^{\infty} \frac{2(-1)^m\sqrt{\frac{2\lambda^3}{K}}\; CF'\sigma_p\, \cos\mu_m y}{b\lambda\mu_m\left[(\lambda-k)^2 - \sigma_m^2\right]}\, e^{\lambda x}$$

$$-\sum_{m=0}^{\infty} \frac{2(-1)^m\sqrt{2\lambda^3/K}\; e^{kx}\, C^2\, \cos\mu_m y}{b\mu_m\left[(2\lambda-k)^2 - \sigma_m^2\right]\sinh 2\sigma_m}$$

$$\times \left[e^{-(2\lambda-k)}\sinh\sigma_m(x-1) - e^{(2\lambda-k)}\sinh\sigma_m(x+1)\right]$$

$$-\sum_{m=0}^{\infty} \frac{2(-1)^m\sqrt{2\lambda^3/K}\; C^2\, \cos\mu_m y}{b\mu_m\left[(2\lambda-k)^2 - \sigma_m^2\right]}\, e^{2\lambda x}$$

$$-\sum_{m=0}^{\infty} \frac{2(-1)^m\sqrt{\lambda^3/2K}\; C\, \cos\mu_m y\, P_1}{b\lambda\mu_m\left[(\lambda+k)^2 - \sigma_m^2\right]}\, F'\sigma_p\, e^{-\lambda x}$$

$$- \sum_{m=0}^{\infty} \frac{2(-1)^m F'\sigma_p \sqrt{\lambda^3/2K}\;\; Ce^{kx} P_1 \cos \mu_m y}{b\lambda\mu_m [(\lambda+k)^2 - \sigma_m^2] \sinh 2\sigma_m}$$

$$\times [e^{(\lambda+k)} \sinh \sigma_m(x-1) - e^{-(\lambda+k)} \sinh \sigma_m(x+1)]$$

$$+ \sum_{m=0}^{\infty} \frac{2(-1)^m C^2 \sqrt{\lambda^3/2K}\;\; e^{kx} P_1 \cos \mu_m y}{b\mu_m^3 \sinh 2\sigma_m}$$

$$\times (e^k \sinh \sigma_m(x-1) - e^{-k} \sinh \sigma_m(x+1))$$

$$+ \sum_{m=0}^{\infty} \frac{2(-1)^m C^3 \sqrt{\lambda^3/2K} \cos \mu_m y\, P_1}{b\mu_m^3} \qquad (5.79)$$

where

$$P_1 = \frac{F'\sigma_p^2}{C^2\lambda^2} + \frac{2k\sigma_p^2 F'}{C^2\lambda}$$

The complete solution $v(x,y)$ of the equation (5.74) is now the sum of the expressions (5.75) and (5.79).

In this case,

$$\text{Volume rate of flow} = F_2(\text{say}) = \int_{-1}^{1}\int_{-b}^{b} v(x,y)\,dx\,dy$$

or $$F_2 = \frac{16b^3}{3} - \frac{16}{b}\sum_{m=0}^{\infty} \frac{\sigma_m(\cosh 2\sigma_m - \cosh 2k)}{\mu_m^6 \sinh \sigma_m \cosh \sigma_m}$$

$$- \sum_{m=0}^{\infty} \frac{8\sqrt{2\lambda^3/K}\, F'\sigma_p C}{b\mu_m^4 \lambda [(\lambda-K)^2 - \sigma_m^2] \sinh 2\sigma_m}$$

$$\times [\sigma_m \cosh(\lambda-2k) + k \sinh 2\sigma_m \sinh\lambda - \sigma_m \cosh 2\sigma_m \cosh \lambda]$$

$$- \sum_{m=0}^{\infty} \frac{8L\sqrt{2\lambda^3/K}\; C^2}{b\mu_m^4\, \lambda\left[(2\lambda-k)^2-\sigma_m^2\right] \sinh 2\sigma_m}$$

$$- \sum_{m=0}^{\infty} \frac{4\sqrt{2\lambda^3/K}\; F'\sigma_p P_1 C}{b\lambda\mu_m^4\left[(\lambda+k)^2-\sigma_m^2\right] \sinh 2\sigma_m}$$

$$\times\left[\sigma_m \cosh(\lambda+2k) - k\sinh 2\sigma_m \sinh\lambda - \sigma_m \cosh 2\sigma_m \cosh\lambda\right]$$

$$+ \sum_{m=0}^{\infty} \frac{4\sqrt{2\lambda^3/K}\; C^2 P_1(\sigma_m \cosh 2k - \sigma_m \cosh 2\sigma_m)}{b\mu_m^6 \sinh 2\sigma_m}$$

$$- \sum_{m=0}^{\infty} \frac{8\sqrt{2\lambda^3/K}\; C F'\sigma_p \sinh\lambda}{b\lambda^2\mu_m^2\left[(\lambda-k)^2-\sigma_m^2\right]}$$

$$- \sum_{m=0}^{\infty} \frac{8\sqrt{2\lambda^3/K}\; C^2 \sinh 2\lambda}{2b\lambda\mu_m^2\left[(2\lambda-k)^2-\sigma_m^2\right]} + \sum_{m=0}^{\infty} \frac{8C^2 P_1\sqrt{\lambda^3/2K}}{b\mu_m^4}$$

$$- \sum_{m=0}^{\infty} \frac{8\sqrt{\lambda^3/2K}\; F'\sigma_p C \sinh\lambda}{b\lambda^2\mu_m^2\left[(\lambda+k)^2-\sigma_m^2\right]} \tag{5.80}$$

The first two terms are same as in case I, giving the volume rate of flow for the particle free fluid, same as obtained by Dean (1959).

### 5.4b Circular cross-section

The solution of the equation (5.59) is given by

$$u(r,\theta) = \int_0^1 \int_{-\pi}^{\pi} \{4e^{kr_o\cos\theta_o} \; G(r,\theta;r_o,\theta_o)\; r_o dr_o d\theta_o$$

$$+ \int_0^1 \int_{-\pi}^{\pi} [\; \frac{C^2\lambda^2(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda r_o\cos\theta_o})e^{(2\lambda-k)r_o\cos\theta_o}}{K^{1/2}[2C^2\lambda e^{2\lambda r_o\cos\theta_o} - 4k\sigma_p^2 F' - 2F'^2\sigma_p^2/\lambda]^{1/2}}$$

$$G(r,\theta;r_o,\theta_o)]\; r_o dr_o d\theta_o \qquad (5.81)$$

The first term in the integrand in the eq. (5.81) gives the solution for the particle free fluid,

using the relations

$$\int_{-\pi}^{\pi} e^{-kr_o\cos\theta_o} \cos n\theta_o d\theta_o = 2\pi(-1)^n I_n(kr_o) \text{ for } n \geq 0 \qquad (5.82)$$

$$\int rI_n(kr)K_n(kr)dr = \frac{r^2}{2}[K_n(kr)\; I''_n(kr) - K'_n\; (kr)\; I'_n(kr)$$

$$+ \frac{1}{rk} K_n(kr)\; I'_n(kr)] \qquad (5.83)$$

$$\int rI_n^2(kr)dr = \frac{r^2}{2}\; [I_n(kr)\; I''_n(kr) - (I'_n(kr))^2$$

$$+ \frac{1}{rk} I'_n(kr)\; I_n(k)] \qquad (5.84)$$

The solution corresponding to the clean gas flow is

$$= \frac{2I_o(kr)\; I'_o(k)}{kI_o(k)} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\theta\; I'_n(kr)\; I_n(kr)}{kI_n(k)}$$

$$- [\frac{2rI'_o(kr)}{k} + 4\sum_{n=1}^{\infty} \frac{(-1)^n\; r\; \cos n\theta\; I'_n(kr)}{k}] \qquad (5.85)$$

The second term in the integrand in the eq. (5.81) gives solution corresponding to the particulate phase. Here, we will solve it for the two cases.

Case I

$$\frac{F'\sigma_p}{\lambda} \gg Ce^{\lambda r\cos\theta}\ , \ \lambda \text{ negative}.$$

The expression

$$\frac{C^2\lambda^2\left(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda r_o\cos\theta_o}\right) e^{(2\lambda-k)r_o\cos\theta_o}}{\left[2C^2\lambda e^{2\lambda r_o\cos\theta_o} - 4k\sigma_p^2F' - 2F'^2\sigma_p^2/\lambda\right]^{1/2}}$$

$$= \frac{\frac{F'}{\lambda} e^{(2\lambda-k)r_o\cos\theta_o} + \frac{C}{\sigma_p} e^{(3\lambda-k)r_o\cos\theta_o}}{\left[-2KF'\ (2k + F'/\lambda)\right]^{1/2}} .$$

Then the solution of the second term in the integrand of the eq. (5.81), after using the eqs. (5.82) to (5.84) is

$$= \frac{C^2\lambda F'\left[I_o(k)I_o((k-2\lambda)r) - I_o(kr)\ I_o(k-2\lambda)\right]}{I_o(k)\ \left[-2KF'(2k+F'/\lambda)\right]^{1/2}\ \left[k^2-(k-2\lambda)^2\right]}$$

$$+ \sum_{n=1}^{\infty} \frac{2C^2\lambda F'(-1)^n\cos n\theta\left[I_n(k)I_n((k-2\lambda)r) - I_n(kr)I_n(k-2\lambda)\right]}{I_n(k)\ \left[-2KF'(2k+F'/\lambda)\right]^{1/2}\ \left[k^2-(k-2\lambda)^2\right]}$$

$$+ \frac{C^3\lambda^2\left[I_o(k)\ I_o((k-3\lambda)r) - I_o(k-3\lambda)\ I_o(kr)\right]}{\sigma_pI_o(k)\ \left[-2KF'(2k+F'/\lambda)\right]^{1/2}\ \left[k^2-(k-3\lambda)^2\right]}$$

$$+\sum_{n=1}^{\infty} \frac{2C^3\lambda^2(-1)^n \cos n\theta \left[ I_n(k)\, I_n((k-3\lambda)r) - I_n(k-3\lambda)\, I_n(kr) \right]}{\sigma_p I_n(k) \left[ -2KF'(2k+F'/\lambda) \right]^{1/2} \left[ k^2-(k-3\lambda)^2 \right]} \tag{5.86}$$

The complete solution $u(r,\theta)$ of the eq. (5.81) is the sum of the expressions (5.85) and (5.86).

It is known that

$$e^{-kr\cos\theta} = I_o(kr) + 2\sum_{n=1}^{\infty} (-1)^n\, I_n(kr)\cos n\theta \tag{5.87}$$

Differentiating (5.87) with respect to k, we get

$$-\, r\cos\theta = rI_o'(kr)\, e^{kr\cos\theta} + 2r\sum_{n=1}^{\infty} (-1)^n\, I_n'(kr)\cos n\theta\, e^{kr\cos\theta} \tag{5.88}$$

Using (5.87) and (5.88), eq. (5.58), with $u(r,\theta)$ given by the sum of the expressions (5.85) and (5.86), gives

$$v = \frac{2r\cos\theta}{k} + \frac{2e^{kr\cos\theta}}{k}\sum_{n=0}^{\infty} \alpha_n I_n(kr)\cos n\theta$$

$$+\frac{C^2\lambda F'\, e^{kr\cos\theta} \left[ e^{(2\lambda-k)r\cos\theta} - \sum_{n=0}^{\infty} \beta_n I_n(kr)\cos n\theta \right]}{\left[ -2KF'\,(2k+F'/\lambda) \right]^{1/2} \left[ k^2-(k-2\lambda)^2 \right]}$$

$$+\frac{C^3\lambda^2\, e^{kr\cos\theta} \left[ e^{(3\lambda-k)r\cos\theta} - \sum_{n=0}^{\infty} \gamma_n I_n(kr)\cos n\theta \right]}{\sigma_p \left[ -2KF'(2k+F'/\lambda) \right]^{1/2} \left[ k^2-(k-3\lambda)^2 \right]} \tag{5.89}$$

where

$$\alpha_o = I_o'(k)/I_o(k)\ , \quad \alpha_n = 2(-1)^n\, I_n'(k)/I_n(k)$$

$$\beta_o = I_o(k-2\lambda)/I_o(k) \ , \ \beta_n = 2(-1)^n \, I_n(k-2\lambda)/I_n(k)$$

$$\gamma_o = I_o(k-3\lambda)/I_o(k) \ , \ \gamma_n = 2(-1)^n \, I_n(k-3\lambda)/I_n(k)$$

Volume rate of flow $F_3(\text{say}) = \int_0^1 \int_{-\pi}^{\pi} v(r,\theta)\, r\,dr\,d\theta$

Using the identities (5.82) to (5.84) and

$$\int_0^1 r I_o(kr)\,dr = \frac{I_1(k)}{k} \tag{5.90}$$

we get

$$F_3 = \sum_{n=0}^{\infty} \Big\{ \frac{2\pi}{k} \alpha_n - \frac{\pi C^2 \lambda F' \beta_n}{[-2KF'(2k + F'/\lambda)]^{1/2} [k^2-(k-2\lambda)^2]}$$

$$- \frac{\pi C^3 \lambda^2 \gamma_n}{\sigma_p [-2KF'(2k + F'/\lambda)]^{1/2} [k^2-(k-3\lambda)^2]} \Big\}$$

$$\times \left[ (1+ \frac{n^2}{k^2}) I_n^2(k) - I_n'^2(k) \right]$$

$$+ \frac{\pi C^2 \lambda F' I_1(2\lambda)}{\lambda [-2KF'(2k+F'/\lambda)]^{1/2} [k^2-(k-2\lambda)^2]}$$

$$+ \frac{2\pi C^3 \lambda^2 I_1(3\lambda)}{3\lambda \sigma_p [-2KF'(2k+F'/\lambda)]^{1/2} [k^2-(k-3\lambda)^2]} \tag{5.91}$$

The first term of the RHS of the eq. (5.91) gives the rate of flow for the particle free fluid, which agrees with the analytical results of Dean (1959).

Case II

$$\frac{F'\sigma_p}{\lambda} \ll Ce^{\lambda r\cos\theta}, \quad C \text{ negative}$$

The expression

$$\frac{C^2\lambda^2\left[\frac{F'\sigma_p}{\lambda} + Ce^{\lambda r_o\cos\theta_o}\right] e^{(2\lambda-k)\, r_o\cos\theta_o}}{\left[2C^2\lambda e^{2\lambda r_o\cos\theta_o} - 4k\sigma_p^2 F' - 2F'^2\sigma_p^2/\lambda\right]^{1/2}}$$

$$= C^2\lambda^2\left(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda r\cos\theta}\right) e^{\lambda r\cos\theta} + \frac{C\lambda^2 P_1}{2(2\lambda)^{1/2}}\left(\frac{F'\sigma_p}{\lambda} + Ce^{\lambda r}\cos\theta\right)$$

$$\times e^{-2\lambda r\cos\theta}$$

With eqs. (5.81) to (5.84), eq. (5.58) gives

$$v = \frac{2r\cos\theta}{k} + \frac{2e^{kr\cos\theta}}{k}\sum_{n=0}^{\infty} \alpha_n I_n(kr)\cos n\theta$$

$$+ \frac{C^2\lambda^2 F'\sigma_p \quad e^{kr\cos\theta}}{[2K\lambda)^{1/2} \quad \lambda[k^2-(k-\lambda)^2]}\left[e^{(\lambda-k)r\cos\theta} - \sum_{n=0}^{\infty} a_n I_n(kr)\cos n\theta\right]$$

$$+ \frac{C^3\lambda^2 e^{kr\cos\theta}}{(2K\lambda)^{1/2}[k^2-(2\lambda-k)^2]}\left[e^{(2\lambda-k)r\cos\theta} - \sum_{n=0}^{\infty} b_n I_n(kr)\cos n\theta\right]$$

$$+ \frac{C\lambda PF'\sigma_p \quad e^{kr\cos\theta}}{2(2K\lambda)^{1/2}[k^2-(2\lambda+k)^2]}\left[e^{(2\lambda+k)r\cos\theta} - \sum_{n=0}^{\infty} c_n I_n(kr)\cos n\theta\right]$$

$$+ \frac{C^2\lambda^2 P_1 e^{kr\cos\theta}}{2(2K\lambda)^{1/2}[k^2-(\lambda+k)^2]}\left[e^{(\lambda+k)r\cos\theta} - \sum_{n=0}^{\infty} d_n I_n(kr)\cos n\theta\right]$$

where

$$a_o = I_o(\lambda - k)/I_o(k), \quad a_n = 2(-1)^n I_n(\lambda - k)/I_n(k)$$

$$b_o = I_o(2\lambda - k)/I_o(k), b_n = 2(-1)^n I_n(2\lambda - k)/I_n(k)$$

$$c_o = I_o(2\lambda + k)/I_o(k), c_n = 2(-1)^n I_n(2\lambda + k)/I_n(k)$$

$$d_o = I_o(\lambda + k)/I_o(k), \quad d_n = 2(-1)^n I_n(\lambda + k)/I_n(k)$$

Volume rate of flow = F4(say) = $\int_0^1 \int_{-\pi}^{\pi} v(r,\theta)\, r\,dr\,d\theta$

$$F_4 = \sum_{n=0}^{\infty} \{\frac{2\pi}{k}\alpha_n - \frac{\pi C^2 \lambda F' \sigma_p a_n}{(2K\lambda)^{1/2}[k^2-(\lambda-k)^2]}$$

$$- \frac{\pi C^3 \lambda^2 \, b_n}{(2K\lambda)^{1/2}[k^2-(2\lambda-k)^2]} - \frac{\pi C \lambda P_1 F' \sigma_p \, c_n}{2(2K\lambda)^{1/2}[k^2-(2\lambda+k)^2]}$$

$$- \frac{\pi C^2 \lambda^2 P_1 \, d_n}{2(2K\lambda)^{1/2}[k^2-(\lambda+k)^2]}\} [(1+\frac{n^2}{k^2})I_n^2(k) - I_n'^2(k)]$$

$$+ \frac{2\pi C^2 F' \sigma_p I_1(\lambda)}{(2K\lambda)^{1/2}[k^2-(\lambda-k)^2]} + \frac{\pi C^3 \lambda \, I_1(2\lambda)}{(2K\lambda)^{1/2}[k^2-(2\lambda-k)^2]}$$

$$+ \frac{\pi C \lambda P_1 F' \sigma_p I_1(2\lambda + 2b)}{2(2K\lambda)^{1/2}(\lambda+k)\,[k^2-(2\lambda+k)^2]}$$

$$+ \frac{\pi C^2 \lambda^2 P_1 I_1(\lambda+2k)}{(2K\lambda)^{1/2}(\lambda+2k)\,[k^2-(\lambda+k)^2]}$$

## 5.5 Discussion of results

In this section, we present the certain basic features of the gas-particulate flow through a curved pipe. For numerical computation, the following values of the parameters are considered:

$k = 0, 1, 2, 3, \ldots, 10$

$c = -.25,\ .25$

$\lambda = -.9,\ .9$

$F' = .05,\ 10,\ 15$

$K = 1.0$

$\sigma_p = .6,\ 1.5$

Fig. 2 gives the variation of the volume flow rate with respect to the secondary flow parameter k, when the secondary motion for the particle phase is a uniform stream. Fig. 2 indicates that as k increases, the volume flow rate decreases. Also, the volume flow rate for the clean gas is more as compared to the flow rate for gas particulate system. This is due to the fact that relaxation time is small, so that the particles are fine.

Fig. 3 gives the variation of the volume flow rate with the secondary flow when the stream lines for the secondary flow of the particle phase are approximated by the exponential curves. Fig. 3 illustrates that in this case also, volume flow rate de

clean gas, volume flow rate is less as compared to the flow rate for gas-particulate system. It is due to the fact that relaxation time in this case is large, so that the particles are coarse enough.

Fig. 4 gives the variation of velocity v with the secondary flow at $y = 0$ for case I. This figure shows that v reduces as the secondary flow through the pipe increases. This figure further indicates that the region where the primary motion is maximum, shifts outwards. Also, the velocity v for the clean gas is more as compared to the gas-particulate flow.

Fig. 5 gives the variation of velocity v with the secondary flow at $y = 0.0$ for case II. This figure indicates that in this case also, v reduces with k and the region, where the primary motion is maximum, shifts outward. Further, the velocity v for gas-particulate flow is more as compared to v for the clean flow.

Figs. 6 and 7 give the variation of volume flow rate and velocity v with the secondary flow parameter k for the case I. For different set of parameters, same trend is observed as in Figs. 2 and 3.

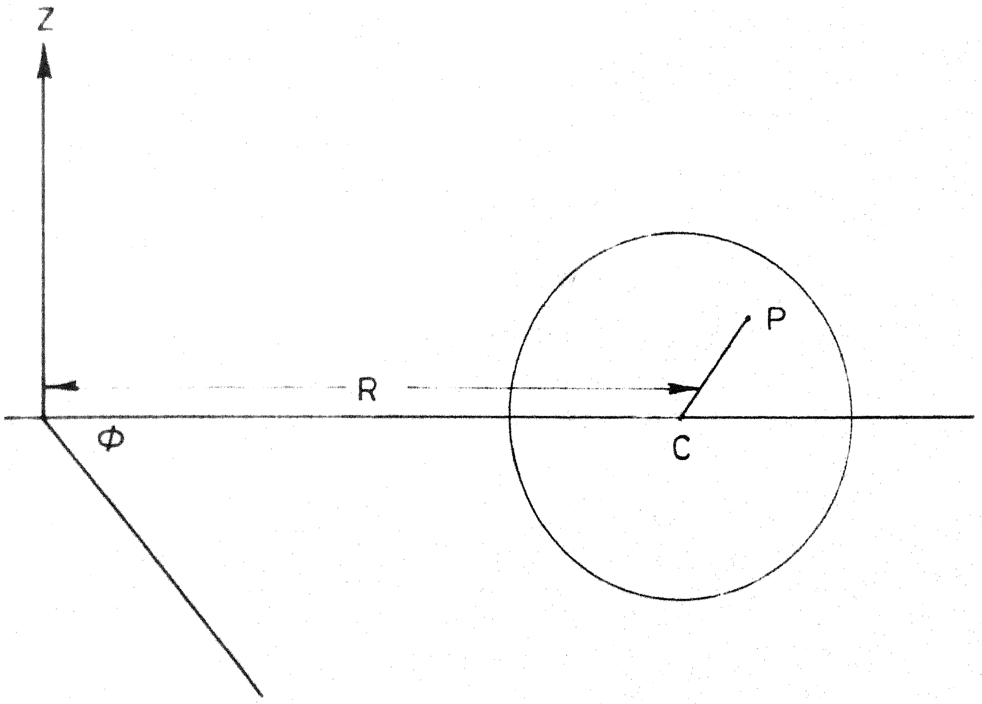



FIG. 1a. COORDINATE SYSTEM

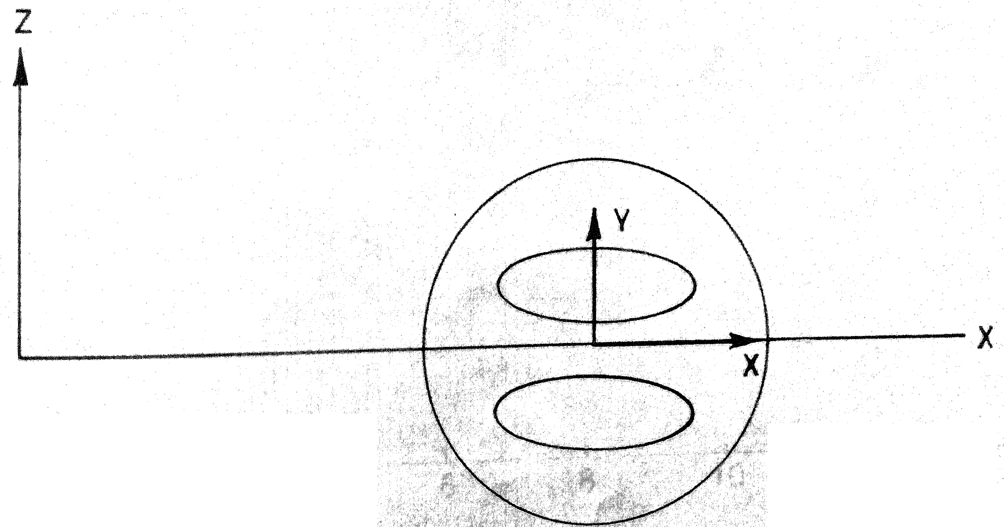



FIG. 1b. TRANSFORMED COORDINATES

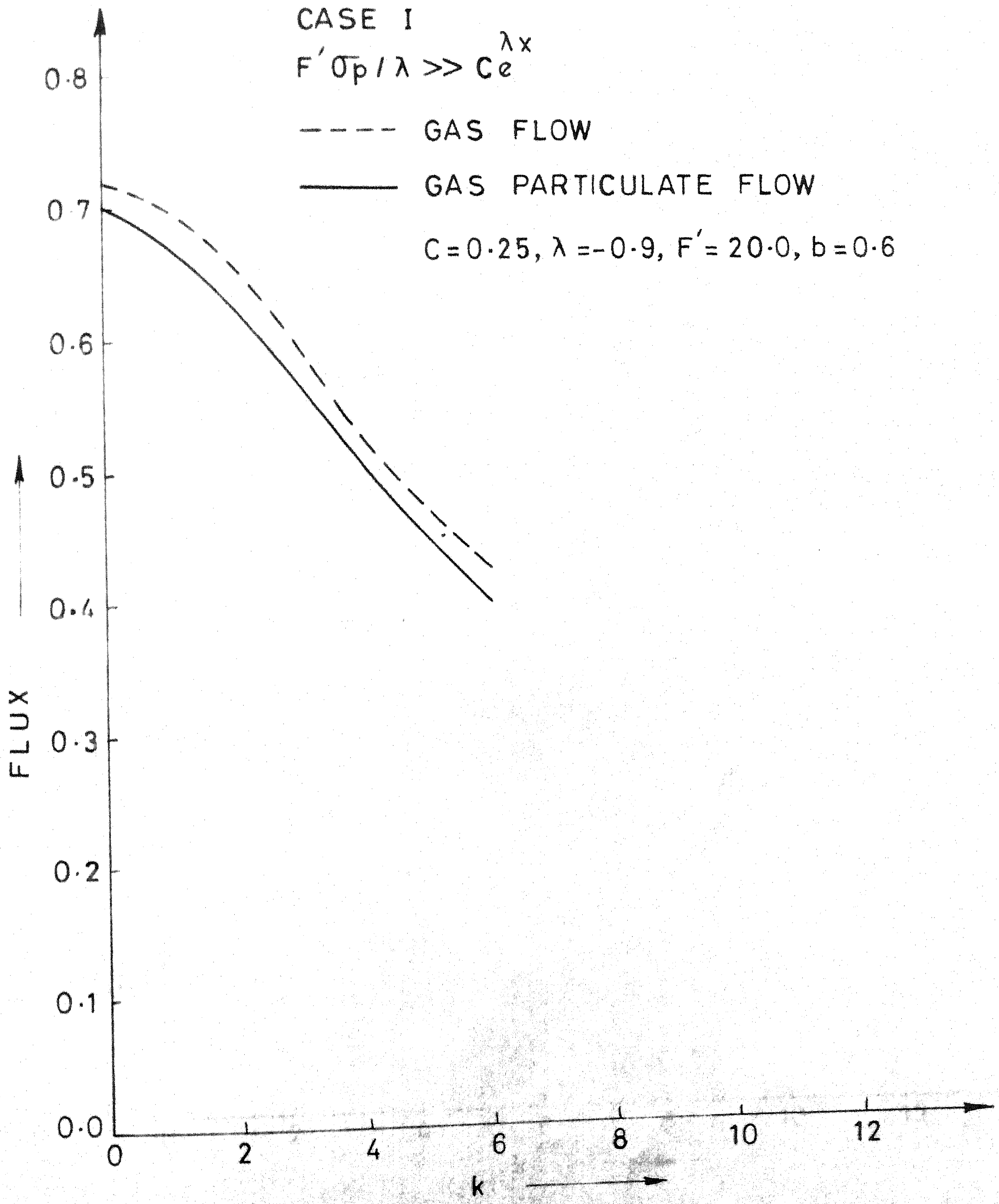



FIG. 2. VARIATION OF VOLUME RATE OF FLOW WITH CHANGES IN k.

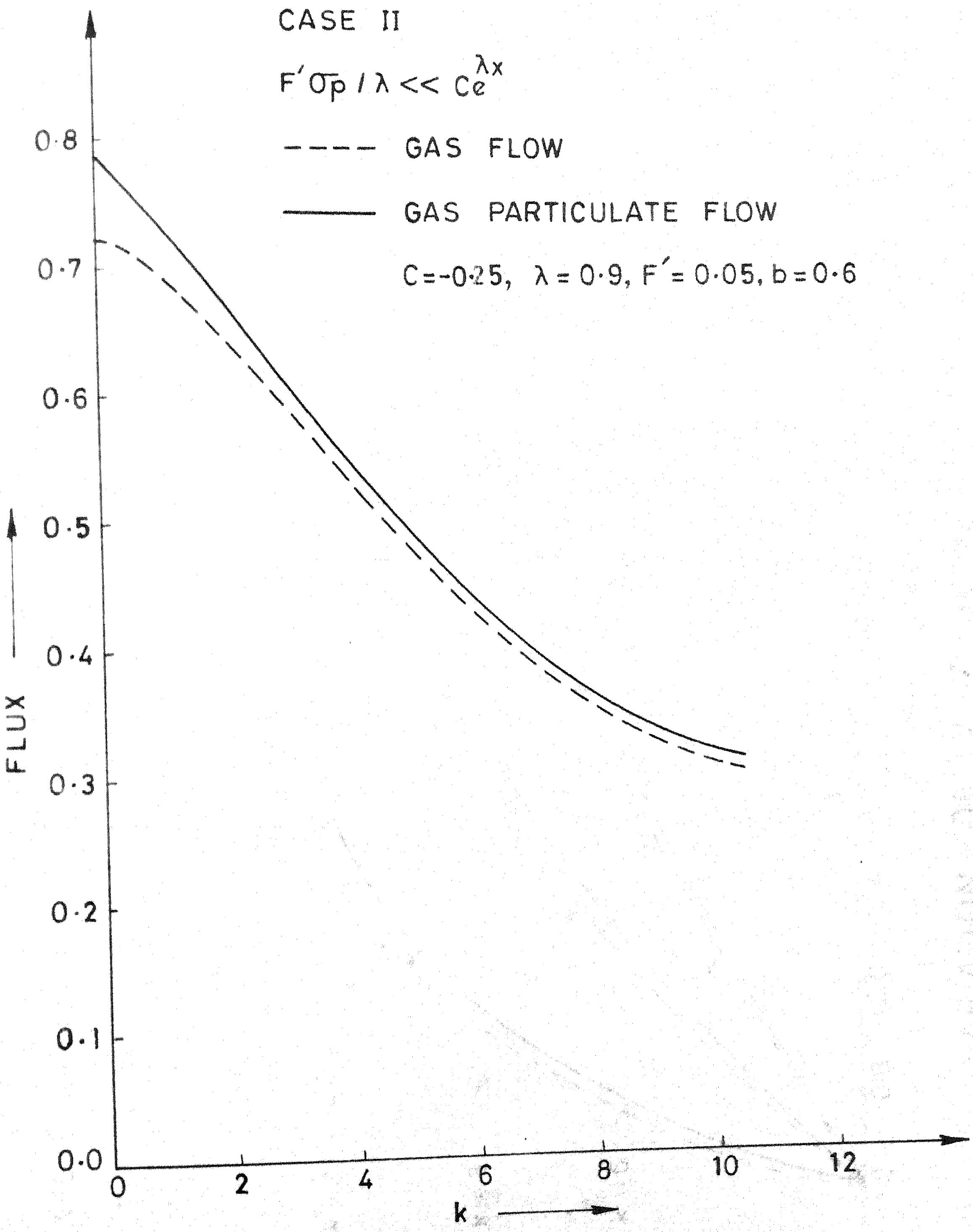



FIG. 3. VARIATION OF VOLUME FLOW RATF WITH CHANGES IN k.

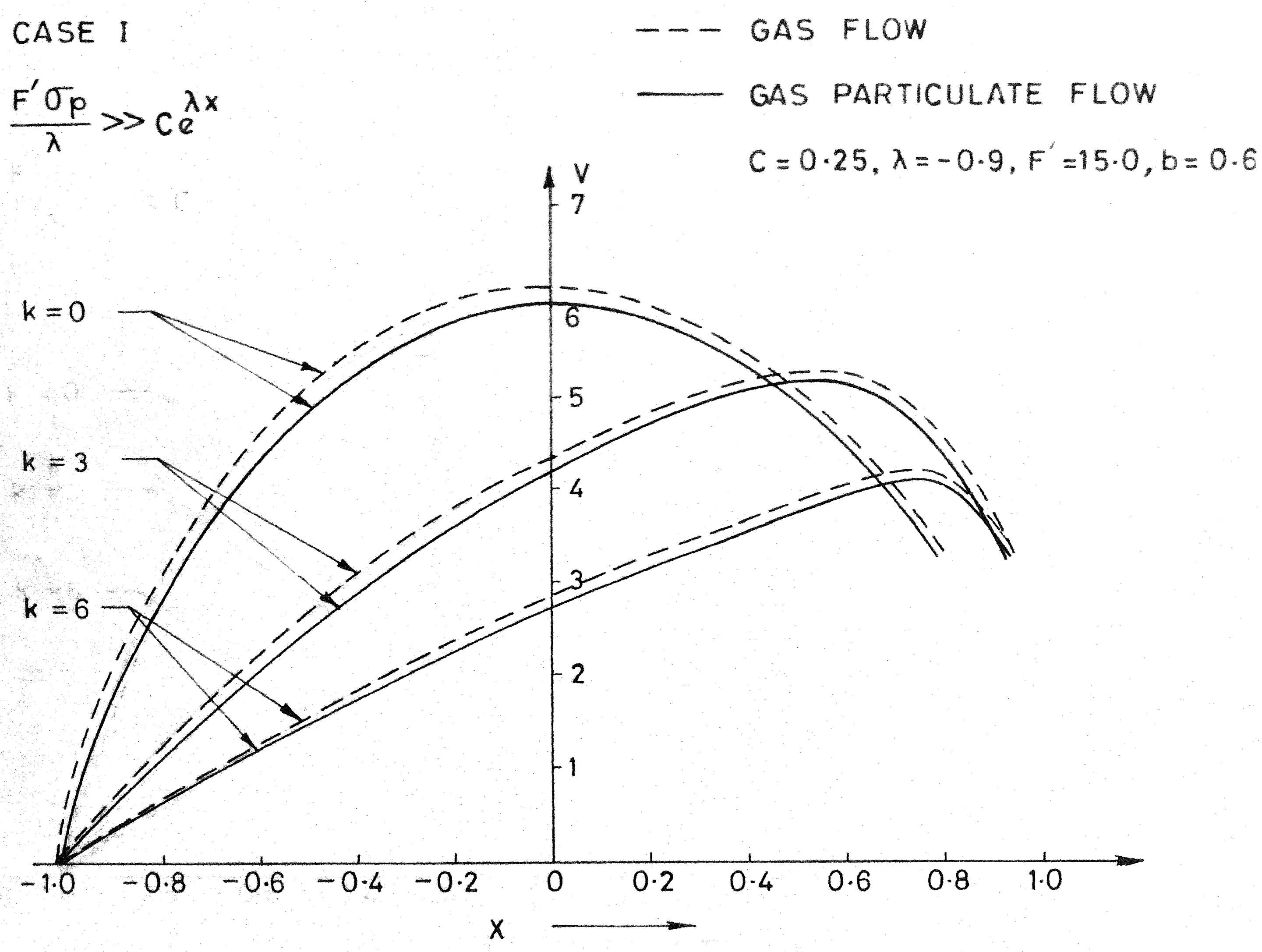



FIG. 4. VARIATION OF V WITH X AT Y = 0·0

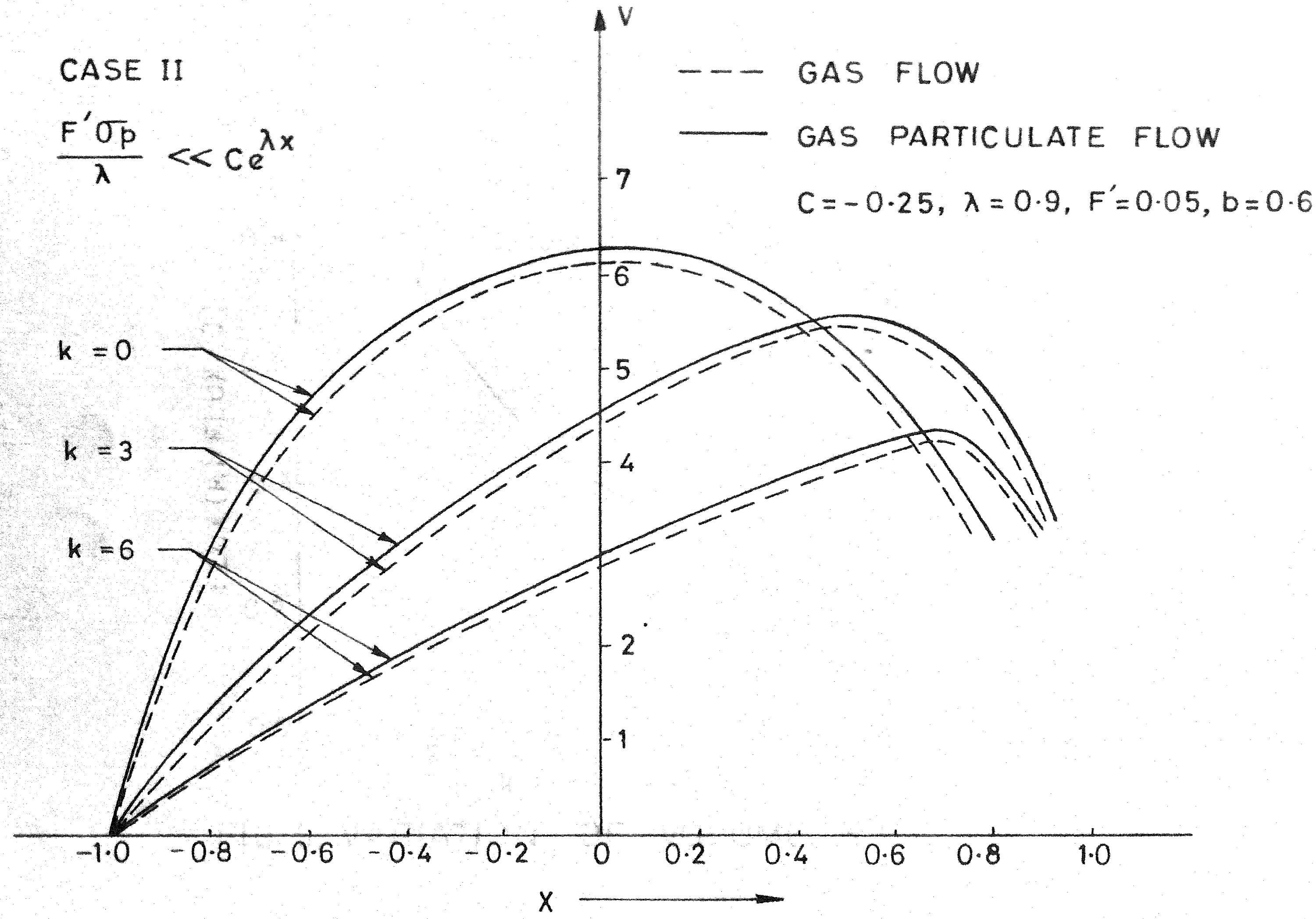



FIG. 5. VARIATION OF V WITH X AT Y = 0·0

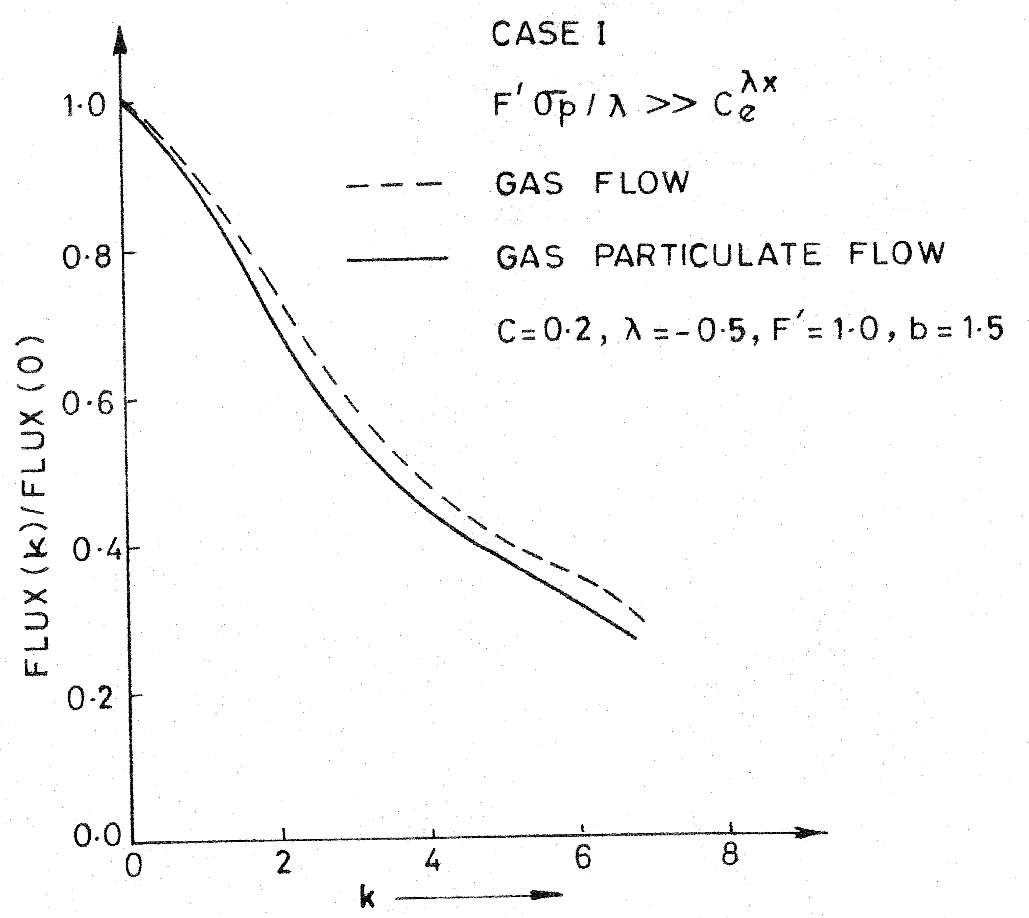



FIG. 6. VARIATION OF VOLUME RATE OF FLOW WITH $k$.

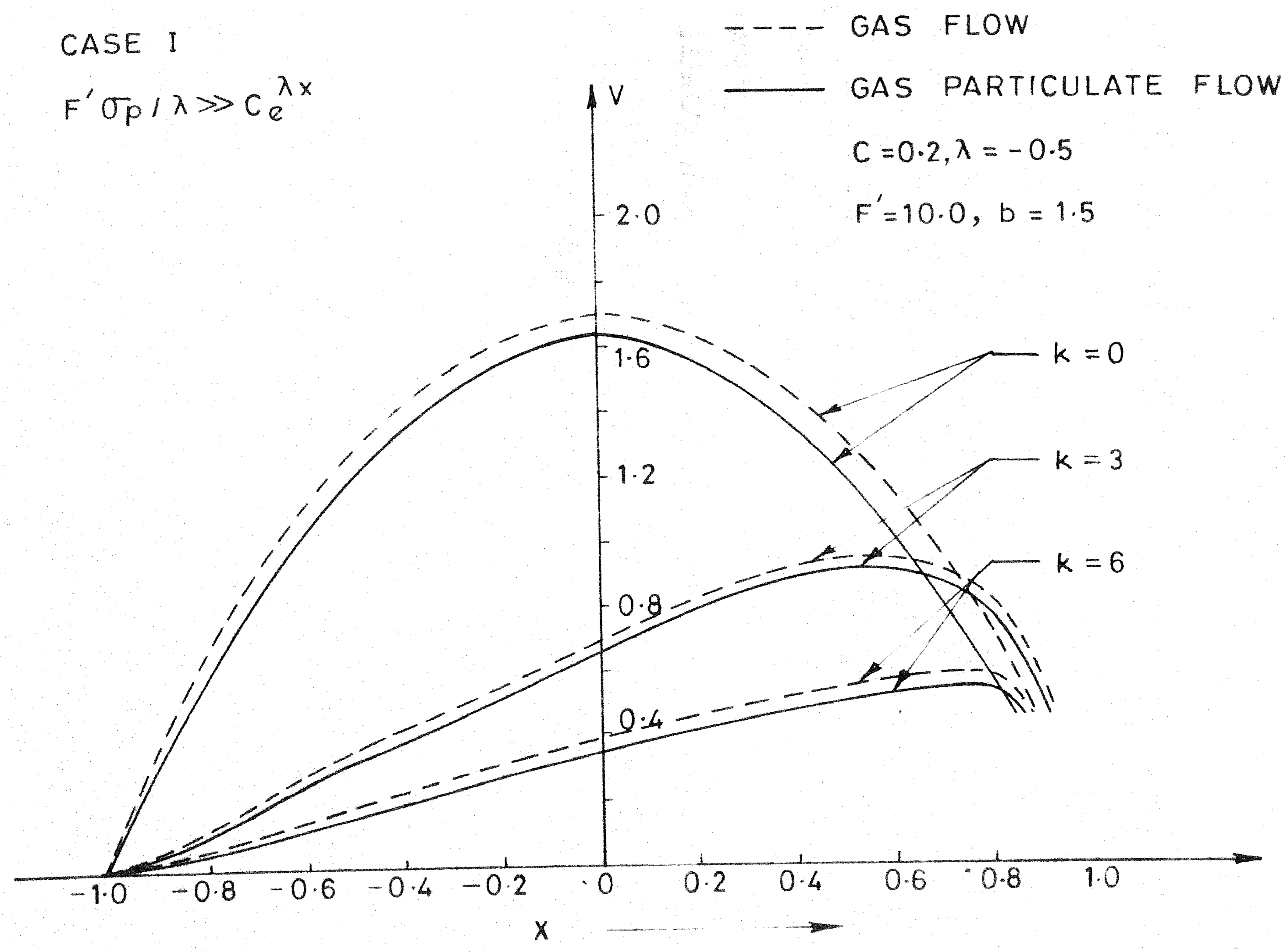



FIG. 7. VARIATION OF V WITH X AT Y = 0·0

# CHAPTER VI

## GLOBAL STABILITY AND UNIQUENESS OF AN INITIAL BOUNDARY VALUE PROBLEM OF A DUSTY GAS MODEL

### 6.1 Introduction

Two types of uniqueness theorem of an initial boundary value problem (IBVP) for Navier-Stokes equations have been pointed out. First of them ensures only one solution for each initial data while the second guarantees only one steady state solution for not too small a viscosity. These results are of significance in hydrodynamic theory of stability.

Foá (1929) found first type of uniqueness theorem for initial value problem. This theorem ensure , uniqueness of regular solution of initial value problem, provided such solutions exist. If solutions develop discontinuities, Foá's theorem may not hold. It is generally supposed that discontinuities can not develop in a viscous fluid but despite great efforts by eminent Mathematicians, it has never been proved [Ladyzhenskaya (1975)]. Serrin (1959) while discussing the concept of universal/Global stability, pointed out the uniqueness of steady solutions for not too small a viscosity. In fact, the two types of uniqueness theorems result as a byproduct of Serrin's concept and analysis of universal stability [Joseph (1976)].

Serrin (1959, 1963) extended the uniqueness theorem to the case of compressible fluids. However, universal stability theorem is not available till today in such a situation.

The purpose of the present chapter is to search for a uniqueness theorem and a theorem for universal stability for a dusty gas model.

There are certain works in this direction. Crooke (1976) examined a uniqueness criterion and Dandapat and Gupta (1976) obtained a universal stability theorem for a dusty gas model. In both the works, the authors considered the constant number density of the particle phase. Dandapat and Gupta (1976) did not obtain any uniqueness criterion while Crooke (1976) did not bother about universal stability theorem. The two works are quite independent.

In the present chapter, we improve the universal stability theorem of Dandapat and Gupta (1976) and as a consequence, we establish a uniqueness criterion. At one point, we depart from the analysis of Dandapat and Gupta to obtain a much improved theorem on universal stability. It ensures a wider range of stability region. We further generalize the analysis to cover the case of non-uniform number density of particle phase. One is able to extract only a uniqueness theorem.

## 6.2 Formulation of the problem

Consider a viscous incompressible two phase flow in a closed bounded volume $\gamma(t)$ whose boundary is $\zeta(t)$. We assume the gas phase to be of constant density while the particle phase to be incompressible with non-uniform number density. The fluid is set in motion by external forces or by the motion of the boundaries. If $\underline{U}(\underline{X},t)$, $\underline{V}(\underline{X},t)$ defines the velocity fields for gas and particle phases respectively, $\pi(\underline{X},t)$ the pressure in $\gamma(t)$ and $N(\underline{X},t)$ the number density of the particles, then basic equations are

$$\frac{\partial \underline{U}}{\partial t} + (\underline{U}.\nabla)\underline{U} - \nu\nabla^2\underline{U} + \nabla\pi - \underline{F}(\underline{X},t) - \frac{KN}{\rho}(\underline{V}-\underline{U}) = 0 \qquad (6.1)$$

$$\text{div } \underline{U} = 0 \qquad (6.2)$$

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V}.\nabla)\underline{V} = \frac{K}{m}(\underline{U}-\underline{V}) + \underline{F}(\underline{X},t) \qquad (6.3)$$

$$\frac{\partial N}{\partial t} + (\underline{V}.\nabla) N = 0 \qquad (6.4)$$

$$\text{div } \underline{V} = 0 \qquad (6.5)$$

Here, $X \in \gamma(t)$, $t > 0$. $\underline{F}(\underline{X},t)$ denotes external force per unit mass, $\nu$ the kinematic viscosity, $\rho_0\pi$ the pressure, m the mass of each particle, K the drag coefficient on the particles. We assume the basic equations to be one as developed by Saffman (1962).

Further, we prescribe the initial and boundary conditions as follows:

$$\underline{U}(\underline{X},t) = \underline{U}_s(\underline{X},t) \quad \underline{X} \in \zeta(t) \quad t \geq 0 \tag{6.6}$$

$$\underline{n} \cdot \underline{V}(\underline{X},t) = \underline{n} \cdot V_s(\underline{X},t) \quad \underline{X} \in \zeta(t) \quad t \geq 0 \tag{6.7}$$

$$U(\underline{X},0) = U_o(X) \quad \underline{X} \in \gamma(0) \tag{6.8}$$

$$V(\underline{X},0) = V_o(X) \quad \underline{X} \in \gamma(0) \tag{6.9}$$

where

$$\operatorname{div} U_o = \operatorname{div} V_o = 0$$

$\gamma(t)$ denotes the volume occupied by the fluid at any instant and $\underline{n}$ the outward drawn unit normal at a point of $\zeta(t)$.

Since the two phases are assumed to be incompressible, total volume remains invariant, however its shape can change. Thus

$$\frac{d}{dt} \int_{\gamma(t)} d\tau = \int_{\gamma(t)} \operatorname{div} \underline{V}(\underline{X},t)\, d\tau = 0$$

where we define

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\underline{V} \cdot \nabla)$$

It is to be noted that material surface is defined with reference to particle phase. Equations (6.1) to (6.9) define an initial boundary value problem (IBVP).

Let $\underline{U}(\underline{X},t,U_o)$ and $\underline{V}(\underline{X},t,V_o)$ denote a solution of (6.1) to (6.9) for a given data $\{\underline{U}_s(\underline{X},t), \underline{V}_s(\underline{X},t), \underline{F}(\underline{X},t), \gamma(t)\}$ for a fixed $\nu$, K,m, $\rho$. The main problem is to establish two theorems of the following type :

<u>Theorem 1</u>. For each $\underline{U}_o$, $\underline{V}_o$, there is only one $\underline{U}(\underline{X},t,\underline{U}_o)$ and $\underline{V}(\underline{X},t,\underline{V}_o)$ satisfying IBVP (6.1) to (6.9).

<u>Theorem 2</u>. If the given data $\{U_s, V_s; F, \gamma\}$ is steady and if $\nu > \bar{\nu}$ then there is atmost one steady solution of IBVP (6.1) to (6.9).

As a byproduct of the stated theorems, we will obtain a global stability criterion. Theorem 2 is essentially significant in the sense that it ensures one steady solution provided kinematic viscosity is large enough. For a smaller viscosity, problem of turbulence arises and for a larger viscosity, fluid may not remain Newtonian and hence basic equation (6.1) may not remain valid.

To prove these two theorems we build relevant structure in the following sections.

## 6.3 <u>Energy equation</u>

Let $\{\underline{U}\ \underline{X},t,\underline{U}_o(X)\}$, $\underline{V}\{\underline{X},t,\underline{V}_o(X)\}$ and $\underline{U}'\{\underline{X},t,\underline{U}_o+\underline{u}_o(X)\}$, $\underline{V}'\{X,t,\underline{V}_o + \underline{v}_o(X)\}$ be two solutions for fixed $\nu, K, m, \rho$ and same data $\{\underline{U}_s(\underline{X},t), \underline{V}_s(X,t), \underline{F}(\underline{X},t), \gamma(t)\}$. Denote

$$\begin{aligned}
\underline{u}(\underline{X},t) &= \underline{U}'(\underline{X},t) - \underline{U}(\underline{X},t)\\
\underline{v}(X,t) &= \underline{V}'(\underline{X},t) - \underline{V}(\underline{X},t)\\
p(\underline{X},t) &= \pi'(\underline{X},t) - \pi(\underline{X},t)\\
n(\underline{X},t) &= N'(\underline{X},t) - N(\underline{X},t)
\end{aligned}$$

Then the basic equations (6.1) to (6.9) reduce to the following:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{U}\cdot\nabla\underline{u}) + \underline{u}\cdot\nabla\underline{U} + \underline{u}\cdot\nabla\underline{u} - \nu\nabla^2\underline{u}$$

$$+ \nabla p - \frac{KN}{\rho}(\underline{v}-\underline{u}) - \frac{Kn}{\rho}(\underline{v}-\underline{u}) - \frac{Kn}{\rho}(\underline{V}-\underline{U}) = 0 \qquad (6.10)$$

$$\operatorname{div}\ \underline{u} = 0 \qquad (6.11)$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{V}\cdot\nabla\underline{v} + \underline{v}\cdot\nabla\underline{V} + \underline{v}\cdot\nabla\underline{v} = \frac{K}{m}(\underline{u}-\underline{v}) \qquad (6.12)$$

$$\operatorname{div}\ \underline{v} = 0 \qquad (6.13)$$

$$\frac{\partial n}{\partial t} + (\underline{v}\cdot\nabla)n + (\underline{v}\cdot\nabla)N + (\underline{V}\cdot\nabla)n = 0 \qquad (6.14)$$

$$\underline{u}(\underline{X},t) = \underline{n}\cdot\underline{v}(\underline{X}.t) = 0 \quad \underline{X} \in \ \zeta(t),\ t \geq 0 \qquad (6.15)$$

$$\underline{u}(\underline{X},0) = \underline{u}_0(X) \quad X \in \ \gamma(0) \qquad (6.16)$$

$$\underline{v}(\underline{X},0) = \underline{v}_0(X) \quad X \in \ \gamma(0) \qquad (6.17)$$

Define

$$E(t) = \frac{1}{2} < |\underline{u}|^2 > = \frac{1}{2}\int_{\gamma(t)} |\underline{u}|^2 d\tau$$

$$e(t) = \frac{1}{2} < |\underline{v}|^2 > = \frac{1}{2}\int_{\gamma(t)} |\underline{v}|^2 d\tau$$

It is to be noted that

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{\gamma(t)} \frac{d}{dt}(|\underline{u}|^2)d\tau + \frac{1}{2}\int_{\gamma(t)} |\underline{u}|^2 \operatorname{div}\ \underline{V}d\tau$$

$$= \frac{1}{2}\int_{\gamma(t)} \frac{\partial}{\partial t}|\underline{u}|^2 d\tau + \frac{1}{2}\int_{\gamma(t)} V_j \frac{\partial}{\partial x_j}|\underline{u}|^2 d\tau + \frac{1}{2}\int_{\gamma(t)} |\underline{u}|^2 \frac{\partial V_j}{\partial x_j}\, d\tau$$

$$= \frac{1}{2}\int_{\gamma(t)} \frac{\partial}{\partial t}|\underline{u}|^2 d\tau + \frac{1}{2}\int_{\gamma(t)} \frac{\partial}{\partial x_j}(V_j|\underline{v}|^2)d\tau$$

$$= \frac{1}{2}\int_{\gamma(t)} \frac{\partial}{\partial t}|\underline{u}|^2 d\tau + \frac{1}{2}\int_{\zeta(t)} n_j\cdot V_j\cdot|\underline{u}|^2 dS$$

$$= \int_{\gamma(t)} \underline{u}\ \frac{\partial \underline{u}}{\partial t}\, d\tau \ .$$

The last integral vanishes owing to first condition of (6.15).

Further

$$\frac{d}{dt}\, e(t) = \frac{1}{2} \int_{\gamma(t)} \frac{d}{dt} |\underline{v}|^2 d\tau + \frac{1}{2} \int_{\gamma(t)} |\underline{v}|^2 \mathrm{div}\, \underline{V}\, d\tau$$

$$= \int_{\gamma(t)} \underline{v}\, \frac{d\underline{v}}{dt}\, d\tau$$

$$= \int_{\gamma(t)} \underline{v}\, (\frac{\partial}{\partial t}\, \underline{v} + \underline{V}.\nabla\underline{v}) d\tau$$

Thus

$$\frac{d}{dt}\, E(t) = \int_{\gamma(t)} \underline{u} \cdot \frac{\partial \underline{u}}{\partial t}\, d\tau \tag{6.18}$$

and

$$\frac{d}{dt}\, e(t) = \int_{\gamma(t)} \underline{v} \cdot (\frac{\partial}{\partial t}\, \underline{v} + \underline{V}.\nabla\underline{v}) d\tau \tag{6.19}$$

Making use of eqs. (6.10) and (6.11), one therefore obtains

$$\frac{d}{dt}\, E(t) = \int_{\gamma(t)} \Big[ \nu u_i \frac{\partial^2 u_i}{\partial x_j^2} - u_i \frac{\partial p}{\partial x_i} - u_i u_j \frac{\partial u_i}{\partial x_j}$$

$$-u_i u_j \frac{\partial U_i}{\partial x_j} - u_i\, U_j \frac{\partial u_i}{\partial x_j}$$

$$+ \frac{KN}{\rho} (u_i v_i - |\underline{u}|^2) + \frac{K}{\rho}\, n (u_i v_i - |\underline{u}|^2)$$

$$+ \frac{Kn}{\rho} (u_i V_i - u_i U_i) \Big] d\tau$$

$$= \int_{\gamma(t)} \frac{\partial}{\partial x_i} \Big[ -p u_i - \frac{1}{2}\, u_j |\underline{u}|^2 - \frac{1}{2}\, U_j |\underline{u}|^2 \Big] d\tau$$

$$- \int_{\gamma(t)} \nu (\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}) d\tau - \int_{\gamma(t)} D_{ij} u_i u_j d\tau$$

$$+ \frac{K}{\rho} \int_{\gamma(t)} (N+n)(u_i v_i - |\underline{u}|^2) d\tau + \frac{K}{\rho} \int_{\gamma(t)} n(u_i V_i - u_i U_i) d\tau$$

where

$$D_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Use of boundary conditions on $\underline{v}$ and Gauss theorem yields

$$\frac{d}{dt} E(t) = - \int_{\gamma(t)} \left[ D_{ij} u_i u_j + \nu \left(\frac{\partial u_i}{\partial x_j}\right)^2 \right] d\tau$$

$$+ \frac{K}{\rho} \int_{\gamma(t)} n \, (u_i V_i - u_i U_i) d\tau$$

$$+ \frac{K}{\rho} \int_{\gamma(t)} (N+n)(u_i v_i - |\underline{u}|^2 d\tau \qquad (6.20)$$

If particle density $N$ happens to be a constant so that $n = 0$, it reduces to

$$\frac{d}{dt} E(t) = - \int_{\gamma(t)} \left[ D_{ij} u_i u_j + \nu \left(\frac{\partial u_i}{\partial x_j}\right)^2 \right] d\tau$$

$$+ \frac{KN}{\rho} \int_{\gamma(t)} (u_i v_i - |\underline{u}|^2) d\tau \qquad (6.21)$$

Similarly, if use is made of eqs. (6.12) and (6.13) then eq. (6.19) gives

$$\frac{d}{dt} e(t) = \int_{\gamma(t)} \underline{v} \cdot \left[ -\underline{v} \cdot \nabla \underline{V} - \underline{v} \cdot \nabla \underline{v} + \frac{K}{m} (\underline{u} - \underline{v}) \right] d\tau$$

$$= - \int_{\gamma(t)} \left( v_i v_j \frac{\partial}{\partial x_j} V_i + v_i v_j \frac{\partial}{\partial x_j} v_i \right) d\tau$$

$$+ \frac{K}{m} \int_{\gamma(t)} (u_i v_i - |\underline{v}|^2) d\tau$$

$$= -\int_{\gamma(t)} \frac{\partial}{\partial x_j}\left(\frac{1}{2} v_j n^2\right) d\tau - \int_{\gamma(t)} n\, v_j \frac{\partial N}{\partial x_j}\, d\tau$$

$$= -\int_{\gamma(t)} n v_j \frac{\partial N}{\partial x_j}\, d\tau$$

The first integral vanishes with the help of Gauss theorem and the second of the boundary conditions (6.15). Thus

$$\frac{d}{dt}\int_{\gamma(t)} \frac{1}{2} n^2\, d\tau = -\int_{\gamma(t)} n\, v_j \frac{\partial N}{\partial x_j}\, d\tau \qquad (6.23)$$

## 6.4 Auxiliary results

In this section, we state/obtain few subsidiary results.

Lemma 1. If $\hat{D}_m < 0$ is the smallest of the three eigen-values of $[D_{ij}]$ over $\gamma(t)$, then

$$-\langle \underline{u}.\underline{\underline{D}}.\underline{u}\rangle = |\hat{D}_m| \leq |\underline{u}|^2\rangle$$

Proof. $\underline{\underline{D}}(\underline{X},t')$, $0 \leq t' \leq t$ is a symmetric matrix defined over $\gamma(t')$. For each fixed $t'$ and at a point $\underline{X}$ of $\gamma(t')$, let $D_{11}(X,t')$, $D_{22}(X,t')$, $D_{33}(X,t')$ be the real eigen values of $\underline{\underline{D}}(\underline{X},t')$. But since

$$T_r\, \underline{\underline{D}} = \text{div}\, \underline{U} = D_{11} + D_{22} + D_{33} = 0$$

therefore, one of the eigenvalues is necessarily negative.

Let $\hat{D}_m < 0$ denote the smallest of the three eigen values over $\gamma(t')$ upto time $t$. Then

$$\hat{D}_m = \min_{\substack{\gamma(t') \\ 0 \leq t' \leq t}} (D_{11}, D_{22}, D_{33})$$

Now at a fixed t' and a fixed point $\underline{X}$ of $\gamma(t')$,

$$\underline{u}\cdot\underline{\underline{D}}\cdot\underline{u} = \underline{u}\cdot\underline{\underline{M}}\,\underline{\underline{D}}_d\,\underline{\underline{M}}^T\cdot\underline{u}$$

where $\underline{\underline{D}}_d$ is diagonal matrix and $M$ is an orthogonal matrix. Thus

$$\underline{u}\cdot\underline{\underline{D}}\cdot\underline{u} = D_{11}V_1^2 + D_{22}V_2^2 + D_{33}V_3^2$$

$$\geq |\underline{V}|^2 \min(D_{11}, D_{22}, D_{33}) \geq D_m|\underline{u}|^2$$

where

$$\underline{V} = \underline{\underline{M}}^T\cdot\underline{u}$$

and

$$|\underline{V}|^2 = |\underline{u}|^2$$

Thus

$$-\langle\underline{u}\cdot\underline{\underline{D}}\cdot\underline{u}\rangle \leq |\hat{D}_m| \langle|\underline{u}|^2\rangle \qquad (6.24)$$

This completes the proof of lemma 1.

Similarly if $\hat{d}_m < 0$ is the smallest of the three eigen values of $[d_{ij}]$ over $\gamma(t)$ upto time t, then

$$-\langle\underline{v}\cdot\underline{\underline{d}}\cdot\underline{v}\rangle \leq |\hat{d}_m| \langle|\underline{v}|^2\rangle \qquad (6.25)$$

It follows from the fact that div. $\underline{V} = 0$

Lemma 2. (Poincare inequality) If $\underline{u}(\underline{X},t)$ is any smooth function such that $\underline{u}(\underline{X},t) = 0 : X \in \zeta(t)$ and div $\underline{u} = 0$, there exists $\Lambda > 2$ such that

$$\frac{L^2}{\Lambda}\langle|\nabla\underline{u}|^2\rangle \geq \langle|\underline{u}|^2\rangle \qquad (6.26)$$

where $\ell(t')$ is the smallest distance between two parallel planes which just contain $\gamma(t)$ and $L = \max\limits_{0 \le t' \le t} \ell(t')$.

One may be referred to Joseph (1976) for the proof of lemma 2.

Use of lemma 1 and lemma 2 reduces energy equations (6.20) to (6.22) to differential inequalities:

$$\frac{d}{dt} E(t) \le 2 \left( |\hat{D}_m| - \frac{\Lambda\nu}{L^2} \right) E(t)$$

$$+ \frac{K}{\rho} \int_{\gamma(t)} n u_i (V_i' - U_i') d\tau$$

$$+ \frac{K}{\rho} \int_{\gamma(t)} N(u_i v_i - |\underline{u}|^2) d\tau \qquad (6.27)$$

For a constant number density

$$\frac{d}{dt} E(t) \le 2 \left( |\hat{D}_m| - \frac{\Lambda\nu}{L^2} \right) E(t) + \frac{KN}{\rho} \int_{\gamma(t)} (u_i v_i - |\underline{u}|^2) d\tau \qquad (6.28)$$

and

$$\frac{d}{dt} e(t) \le 2 |\hat{d}_m| e(t) + \frac{K}{m} \int_{\gamma(t)} (u_i v_i - |\underline{v}|^2) d\tau \qquad (6.29)$$

We shall quite often be using estimates of the integrals of the type

$$\int_{\gamma(t)} F(x)\, f_i(x)\, g_i(x) d\tau \text{ , } F(x) > 0 \text{ over } \gamma(t)$$

We can estimate it in two ways.

Let $\qquad \alpha = \max_{\substack{\gamma(t) \\ 0 \leq t \leq t'}} |F(x)|$

Then

$$2 \int_{\gamma(t)} F(x)\ f_i(x)\ g_i(x) d\tau \leq \int_{\gamma(t)} F(x)\ (f_i^2 + g_i^2) d\tau$$

$$\leq \alpha \int_{\gamma(t)} (f_i^2 + g_i^2) d\tau$$

Thus

$$\int_{\gamma(t)} F(x)\ f_i(x)\ g_i(x) d\tau \leq \alpha \left(\frac{1}{2} \int_{\gamma(t)} f_i^2 d\tau + \frac{1}{2} \int_{\gamma(t)} g_i^2 d\tau\right) \tag{6.30}$$

Further, using Schwarz inequality, one gets

$$\left( \int_{\gamma(t)} F(x)\ f_i(x)\ g_i(x) d\tau\right)^2 \leq \int_{\gamma(t)} F^2(x) f_i^2(x) d\tau \int_{\gamma(t)} g_i^2(x) d\tau$$

$$\leq \alpha^2 \int_{\gamma(t)} f_i^2(x) d\tau \int_{\gamma(t)} g_i^2(x) d\tau$$

Thus

$$\int_{\gamma(t)} F(x)\ f_i(x)\ g_i(x) d\tau \leq 2\alpha \sqrt{\frac{1}{2} \int f_i^2 d\tau} \sqrt{\frac{1}{2} \int g_i^2\ d\tau} \tag{6.31}$$

$$\leq \alpha\left(\frac{1}{2} \int f_i^2 d\tau + \frac{1}{2} \int g_i^2\ d\tau\right) \tag{6.32}$$

In fact, it is obvious from above that (6.30) follows from (6.31). Moreover, we note that $F(x) \geq 0$, is not relevant if (6.32) [the same as (6.30)] is derived from (6.31).

The two estimates are important and we use them in the analysis to follow. We list below certain estimates which are useful in further analysis. These can be derived along the lines indicated above.

$$\int_{\gamma(t)} u_i v_i d\tau \leq e(t) + E(t) \tag{6.33}$$

$$\int_{\gamma(t)} u_i v_i d\tau \leq 2\sqrt{e(t)}\sqrt{E(t)} \tag{6.34}$$

$$\int_{\gamma(t)} n v_j \frac{\partial N}{\partial x_j} d\tau \leq \beta_1(\theta + e) \tag{6.35}$$

$$\int_{\gamma(t)} n u_i (V_i' - U_i') d\tau \leq \beta_2(\theta + E) \tag{6.36}$$

$$\int_{\gamma(t)} N u_i v_i d\tau \leq \beta_3(e + E) \tag{6.37}$$

where

$$\left.\begin{aligned} \theta &= \frac{1}{2}\int_{\gamma(t)} n^2 d\tau, \quad \beta_1 = \max_{\substack{\gamma(t)\\ 0\leq t\leq t'}} \left|-\frac{\partial N}{\partial x_j}\right| \\ \beta_2 &= \max_{\substack{\gamma(t)\\ 0\leq t\leq t'}} |V'-U'|, \quad \beta_3 = \max_{\substack{\gamma(t)\\ 0\leq t\leq t'}} N \\ \beta_4 &= \min_{\substack{\gamma(t)\\ 0\leq t\leq t'}} N \end{aligned}\right] \tag{6.38}$$

## 6.5 Main theorems

The case of variable number density

The relevant equations/inequalities for the present purpose will be (6.23), (6.27) and (6.29). The use of the estimates (6.33) to (6.37) for various integrals yield

$$\frac{d\theta}{dt} \leq \beta_1(\theta + e) \tag{6.39}$$

$$\frac{dE}{dt} \leq \left[2|\hat{D}_m| - \frac{2\Lambda\nu}{L^2} + \frac{K}{\rho}\beta_2 + \frac{K}{\rho}\beta_3 - \frac{2K}{\rho}\beta_4\right]E + \frac{K}{\rho}\beta_2\theta + \frac{K}{\rho}\beta_3 e \tag{6.40}$$

and

$$\frac{de}{dt} \leq (2|\hat{d}_m| - \frac{K}{m})\, e + \frac{K}{m}\, E \tag{6.41}$$

These inequalities yield that

$$\frac{d}{dt}(\theta + E + e) \;\leq\; \beta(\theta + E + e) \tag{6.42}$$

where

$$\beta = \max\Big[2(|\hat{D}_m| - \frac{\Lambda\nu}{L^2}) + \frac{K}{\rho}(\beta_2+\beta_3-2\beta_4) + \frac{K}{m}\,,$$

$$2|\hat{d}_m| - \frac{K}{m} + \frac{K}{\rho}\beta_3 + \beta_1,\; \beta_1 + \frac{K}{\rho}\beta_2\Big]$$

It is trivial to observe that $\beta > 0$.

<u>Theorem 1</u>. Integrals of (6.42) over $[0,t']$ gives

$$\theta(t') + E(t') + e(t') \leq [\theta(0) + E(0) + e(0)]\exp(\beta t')$$

where $E(0)$ and $e(0)$ are given by

$$E(0) = \frac{1}{2}\int_{\gamma(0)} |\underset{\sim}{u}_o|^2\, d\tau$$

$$e(0) = \frac{1}{2}\int_{\gamma(0)} |\underset{\sim}{v}_o|^2\, d\tau$$

and

$$\theta(0) = 0$$

In case $\underset{\sim}{v}_o = 0 = \underset{\sim}{u}_o$, $\theta(t')$, $E(t')$, $e(t')$ will separately vanish since $e(0)$ and $E(0)$ vanish. It implies that only one solution will start from each initial data.

The case of constant number density

Eq. (6.23) will be redundant and only differential inequalities (6.28) and (6.29) will be relevant. Use of estimates (6.33) reduces the inequalities to the form

$$\frac{dE}{dt} \leq (2|\hat{D}_m| - \frac{2\Lambda\nu}{L^2} - \frac{KN}{\rho})\ E(t) + \frac{KN}{\rho}\ e \qquad (6.43)$$

$$\frac{de}{dt} \leq (2|\hat{d}_m| - \frac{K}{m})\ e(t) + \frac{K}{m}\ E \qquad (6.44)$$

It will be noted later that this estimate gives a wider range for stability region as compared to Dandapat and Gupta (1976) who used the estimate as provided by (6.43).

Inequalities (6.43) and (6.44) yield

$$\frac{d}{dt}\ (E+e) \leq \beta_o\ (e+E) \qquad (6.45)$$

where

$$\beta_o = \max\ [2|\hat{D}_m| - \frac{KN}{\rho} - \frac{2\Lambda\nu}{L^2} + \frac{K}{m}\ ,\ 2|\hat{d}_m| - \frac{K}{m} + \frac{KN}{\rho}] \qquad (6.46)$$

and is a constant. Inequality (6.45) holds for all times for which $\beta_o$ is a constant. Proof of Theorem 1 follows exactly along the same lines as for the case of variable number density.

Proof of Theorem 2. Let $\beta_o = -\bar{\beta}$ and $\bar{\beta} > 0$, then inequality (6.45) gives

$$E(t') + e(t') \leq [E(0) + e(0)]\ \exp\ (-\bar{\beta}t')\quad 0 \leq t' \leq t$$

For a steady solution $E(t') = E(0)$ and $e(t') = e(0)$. It

is possible iff $E(t') = e(t') = 0$. Thus only one steady solution is possible.

One fails to prove this result if $\beta_0$ happens to be positive. We note that if both the expressions in the paranthesis of (6.46) are negative, $\bar{\beta}$ will definitely be negative and under the condition only one steady solution will be possible. Thus

$$2|\hat{D}_m| - \frac{KN}{\rho} - \frac{2\Lambda\nu}{L^2} + \frac{K}{m} < 0$$

and

$$2|\hat{d}_m| - \frac{K}{m} + \frac{KN}{\rho} < 0$$

is a sufficient condition for only one steady solution of the system. Alternatively, one might put that

$$2|\hat{d}_m| < K(\frac{1}{M} - \frac{N}{\rho}) < 2\,(\frac{\Lambda\nu}{L^2} - |\hat{D}_m|) \tag{6.47}$$

is a sufficient condition only for one steady solution of the system. It is implicit that the condition is meaningful provided

$$\frac{N}{\rho} < \frac{1}{m} \tag{6.48}$$

and

$$\nu > \frac{L^2}{\Lambda}\,(|\hat{d}_m| + |\hat{D}_m|) \tag{6.49}$$

These conditions can also be regarded as sufficient conditions for stability for the solution U,V. Inequality (6.48) means that mass density of the dust must be less than the density of the gas and (6.49) implies that Kinematic

viscosity must not be too small. Thus for not too small a Kinematic viscosity of gas, having light suspended particles, (6.47) ensures stability.

## 6.6 Improved results

We note that character of $\beta$ in the inequality (6.42) and that of $\beta_0$ in the inequality (6.45) is of basic importance. For uniqueness theorem, it is immaterial whether $\beta$ and $\beta_0$ are positive or negative. However, for stability theorem, their negative character is a must.

Keeping the observations in view, we improve the sufficient conditions as given by (6.47) for the stability of the flow. We attempt to show the existence of $\lambda_1 > 0$ and $\lambda_2 < 0$ such that (6.43) and (6.44) yield

$$\frac{d}{dt}(e + \lambda_1 E) \leq \lambda_2 (e + \lambda_1 E) \tag{6.50}$$

For such $\lambda_1$ and $\lambda_2$, the following must be identically true.

$$\lambda_2 = 2|\hat{d}_m| - \frac{K}{m} + \lambda_1 \frac{KN}{\rho} \tag{6.51}$$

and

$$\lambda_1\lambda_2 = \frac{K}{m} + \lambda_1\left(2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*}\right) \tag{6.52}$$

where

$$\nu^* = \frac{L^2}{\Lambda}$$

The two equations give that $\lambda_2$ must be a root of the quadratic equation

$$\lambda_2^2 - \lambda_2 \left[ 2|\hat{d}_m| + 2|\hat{D}_m| - \frac{K}{m} - \frac{KN}{\rho} - \frac{\nu}{\nu^*} \right]$$

$$+ \left(2|\hat{d}_m| - \frac{K}{m}\right)\left(2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*}\right) - \frac{K^2N}{\rho m} = 0$$

Since $\lambda_1$ is required to be positive and $\lambda_2 < 0$. One of the sufficient conditions for the existence of $\lambda_1 > 0$ and $\lambda_2 < 0$ will be that $2|\hat{d}_m| - \frac{K}{m} < 0$ and a root of the above equation lies between 0 and $2|\hat{d}_m| - \frac{K}{m}$.

Therefore a sufficient condition of stability of the system is that

$$2|\hat{d}_m| - \frac{K}{m} < 0$$

and

$$\left(2|\bar{d}_m| - \frac{K}{m}\right)\left(2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*}\right) - \frac{K^2N}{\rho m} \quad \text{and}$$

$$\left(2|\hat{d}_m| - \frac{K}{m}\right)^2 - \left(2|\hat{d}_m| - \frac{K}{m}\right)\left[2|\hat{d}_m| + |\hat{D}_m| - \frac{K}{m} - \frac{KN}{\rho} - \frac{\nu}{\nu^*}\right]$$

$$- \frac{K^2N}{\rho m} + \left(2|\hat{d}_m| - \frac{K}{m}\right)\left(2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*}\right) \quad \text{are of}$$

opposite signs.

The last of the above expressions is equal to $-\frac{K^2N}{\rho}$ which is negative. Therefore a sufficient condition for stability is that

$$2|\hat{d}_m| - \frac{K}{m} < 0$$

and $\left(2|\hat{d}_m| - \frac{K}{m}\right)\left(2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*}\right) - \frac{K^2N}{\rho m} > 0$

It implies that the flow is stable if

$$2|\hat{d}_m| - \frac{K}{m} < 0$$

and

$$2|\hat{d}_m|(\frac{KN}{\rho} - 2|\hat{D}_m| + \frac{\nu}{\nu^*}) < \frac{K}{m}(\frac{\nu}{\nu^*} - 2|\hat{D}_m|)$$

we note that the condition fails if

$$2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*} > 0$$

we have thus established the following theorem.

<u>Theorem</u>. A sufficient condition for stability of a dusty gas flow is that

$$2|\hat{d}_m| - \frac{K}{m} < 0 \qquad (6.55)$$

$$2|\hat{D}_m| - \frac{KN}{\rho} - \frac{\nu}{\nu^*} < 0 \qquad (6.54)$$

and

$$2|\hat{d}_m|(\frac{KN}{\rho} - 2|\hat{D}_m| + \frac{\nu}{\nu^*}) < \frac{K}{m}(\frac{\nu}{\nu^*} - 2|\hat{D}_m|) \qquad (6.55)$$

A question arises: Could the analysis developed in this section lead to a sufficiency theorem for the case of a variable number density? Let if possible, there exists $C_1 > 0$, $C_2 > 0$ and $C_3 < 0$ in such a way that the inequalities (6.39) to (6.41) yield

$$\frac{d}{dt}(E + C_1 e + C_2 \theta) \leq C_3 (E + C_1 e + C_2 \theta) \qquad (6.56)$$

This differential inequality gives

$$C_3 = [2|\hat{D}_m| - \frac{2\Lambda\nu}{L^2} + \frac{K}{\rho}(\beta_2 + \beta_3 - 2\beta_4) + \frac{K}{m} C_1$$

$$C_1 C_3 = \frac{K\beta_3}{\rho} + C_1(2|\hat{d}_m| - \frac{K}{m}) + C_2\beta_1$$

$$C_2 C_3 = \frac{K}{\rho}\beta_2 + C_2\beta_1 .$$

The last of the equations imply that $C_2(C_3-\beta_1) = \frac{K}{\rho}\beta_2$. For $C_2$ to be positive, one requires that $C_3 > \beta_1$. $\beta_1$ and $\beta_2$ being positive $C_3$ will consequently be positive. It shows that $C_2 > 0$ and $C_3 < 0$ are not consistent. Present analysis therefore does not yield a stability theorem.

## 6.7 A comparison of stability theorems

For a comparative study of stability theorem, it is desirable to rederive Dandapat and Gupta's stability theorem.

Let us denote $E = K_1^2$ and $C = K_2^2$. Then use of the estimate as given by (6.34) for the integral $\int u_i v_i d\tau$ reduce the inequalities (6.28) and (6.29) to the forms

$$\frac{d}{dt} K_1 \leq \left( |\hat{D}_m| - \frac{\Lambda\nu}{L^2} - \frac{KN}{\rho}\right) K_1 + \frac{KN}{\rho} K_2$$

and

$$\frac{d}{dt} K_2 \leq \frac{K}{m} K_1 + \left(|\hat{d}_m| - \frac{K}{m}\right) K_2$$

The two inequalities imply that

$$\frac{d}{dt}(K_1 + K_2) \leq \gamma(K_1 + K_2)$$

where

$$\gamma = \max \left[ |\hat{D}_m| - \frac{\Lambda\nu}{L^2} - \frac{KN}{\rho} + \frac{K}{m} \,,\; |\hat{d}_m| - \frac{K}{m} + \frac{KN}{\rho} \right]$$

system is therefore stable provided

$$|\hat{D}_m| - \frac{\Lambda\nu}{L^2} - \frac{KN}{\rho} + \frac{K}{m} < 0 \quad \text{and} \quad |\hat{d}_m| - \frac{K}{m} + \frac{KN}{\rho} < 0$$

Theorem. A sufficient condition for the stability of two phase flow for a constant number density case is that

$$|\hat{d}_m| < K(\frac{1}{m} - \frac{N}{\rho}) < (\frac{\Lambda\nu}{L^2} - |\hat{D}_m|) \qquad (6.57)$$

It is also desirable to note the following theorem.

Theorem. A sufficient condition for the stability of single phase fluid flow is that

$$\frac{\Lambda\nu}{L^2} - |\hat{D}_m| > 0 \qquad (6.58)$$

We will now compare the sufficiency theorems for stability of two phase flow as provided by the inequalities (6.47), (6.57) and by the inequalities (6.53) to (6.55).

We assume single phase fluid flow to be stable satisfying the condition (6.58) and particle mass density to be smaller than that of gas density so that $\frac{1}{m} - \frac{N}{\rho} > 0$. We are therefore interpreting the results for light dust with a base flow which in the absence of dust is stable. The shaded regions in [Fig.1] $(\frac{K}{m}, \frac{KN}{\rho})$ plane indicate stable regions. Dotted lines, broken lines and solid lines bound regions which are respectively provided by Dandapat and Gupta and by the present author [(6.47), (6.53) to (6.55)].

It is clear that sufficiency region obtained by the present author is much enlarged region.

Since for the light dust, we have not shown stable base flow to be always stable , **the** present analysis does hint that light dust is destabilising in nature. However, it can not be regarded as a conclusive statement as theorems obtained are only sufficiency theorems whose violation does not necessarily mean instability.

## 6.8 Conclusions

For a constant number density two phase flow, we are able to delineate stability regions for light dust with stable base flow. This region is much enlarged as compared to the one obtained by Dandapat and Gupta.

A uniqueness theorem for certain initial boundary value problem for a constant as well as variable number density case has been established. It has also been established that the present analysis will not yield a sufficiency theorem for stability for variable number density case of two phase flow.

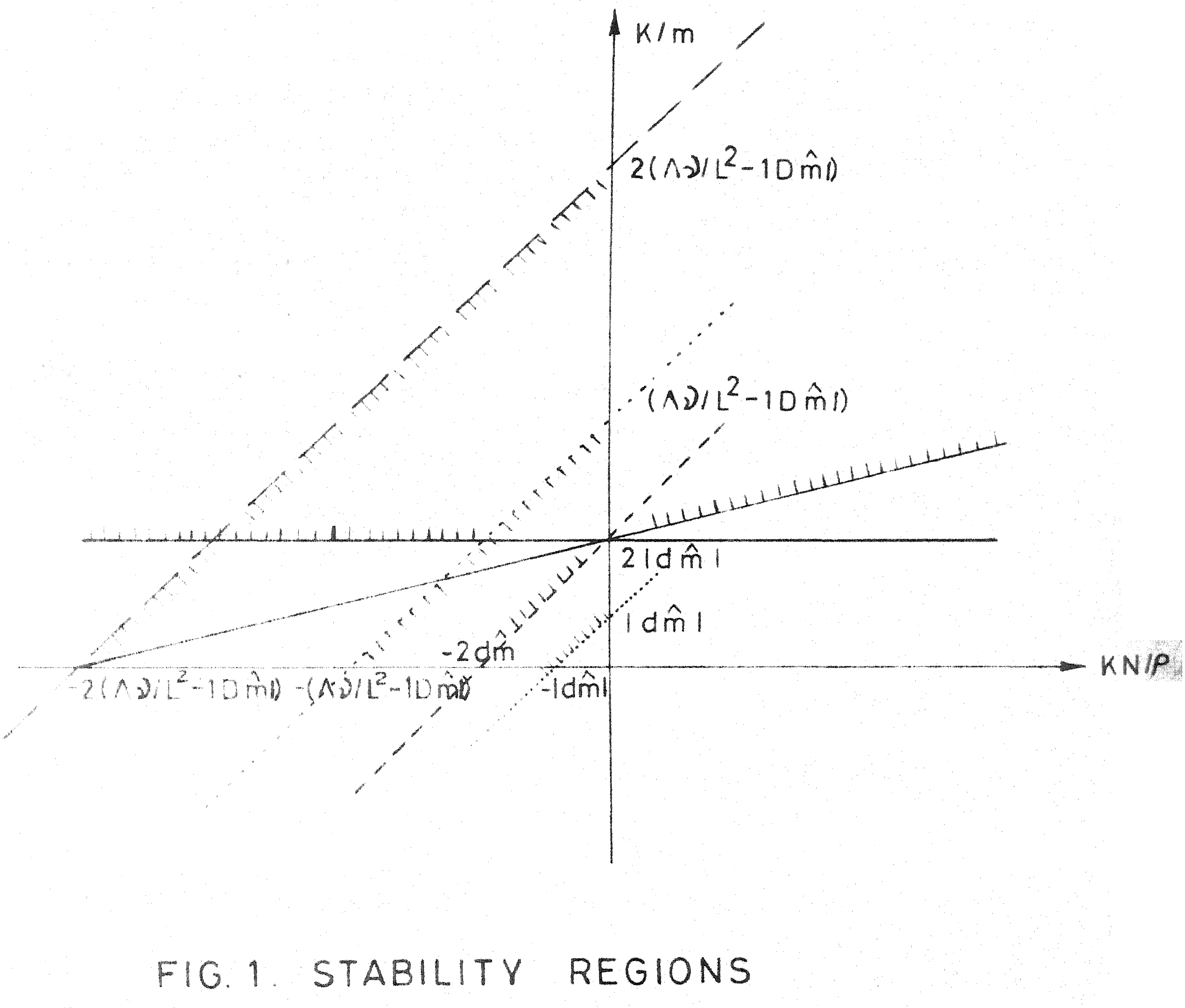



FIG. 1. STABILITY REGIONS

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